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Stability analysis methods and their applicability to car-following models: review and comparison

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Abstract: The paper comprehensively reviews major methods for analysing local and string stability of car-following (CF) models. Specifically, three types of CF models are considered: basic, time-delayed, and multi-anticipative/ cooperative CF models. For each type, notable methods in the literature for analysing its local stability and string stability have been reviewed in detail, including the characteristic equation based method (e.g., root extracting, the root locus method, the Routh-Hurwitz criterion, the Nyquist criterion and the Hopf bifurcation method), Lyapunov criterion, the direct transfer function based method, and the Laplace transform based method. In addition, consistency and applicability of stability criteria obtained using some of these methods are objectively compared with the simulation result from a series of numerical experiments. Finally, issues, challenges, and research needs of CF models' stability analysis in the era of connected and autonomous vehicles are discussed.

Keywords: Car following; stability analysis; numerical experiment; connected and autonomous vehicles; IDM

1. Introduction

As a typical feature of traffic congestion, stop/slow-and-go oscillations are a main negative externality of road transportation systems, often triggered by instability in the car following (CF) behaviour (Chandler et al., 1958; Wilson and Ward, 2011; Zheng et al., 2011). The growth of perturbation (e.g. a spacing and speed deviation from a steady state) over time and space during CF leads to the instability of traffic flow. In general, the aim of stability analysis is to study how the perturbation (perturbation and disturbance are used interchangeably in this paper) of a leading vehicle evolves over time and space using CF models by assuming that vehicles travel on a single lane without overtaking.

Broadly speaking, there are two types of stability analysis: linear stability analysis and nonlinear stability analysis. Linear stability analysis focuses on stability characteristics of a system under the influence of a small perturbation, while nonlinear stability analysis on stability characteristics of a system under the influence of a large perturbation. For two main reasons, linear stability analysis has been the theme of the majority of the literature in traffic flow theories where nonlinear CF system is often linearized around the equilibrium point: i) for road traffic, disturbances experienced by road users are often small; and ii) nonlinear stability analysis is much more complicated than linear stability analysis. Thus, linear stability analysis is the focus of this paper, too. Before any further discussion, some confusion on terminologies used in the literature needs to be clarified. The fact that stability analysis is an important topic, and thus has been investigated in different disciplines (e.g., numerical analysis, control theory, dynamic systems, and of course traffic flow modelling), has caused confusions in the literature in terms of terminologies. For example, different disciplines may define/classify stabilities differently, as pointed out in Zhang and Jarrett (1997) and Treiber and Kesting (2013). In dynamic systems, only local stability is studied, which is categorized as Lyapunov stability (any sufficiently small initial perturbation always remains small) and asymptotic stability (any sufficiently small initial perturbation tends to zero as time approaches infinity). In traffic flow modelling, two types of stability have also been investigated but defined differently: local stability is to investigate the stability of a single vehicle's movement over time under the influence of a small perturbation that is often originated from the leading vehicle's movement (It is locally stable if the perturbation diminishes over time, which corresponds to asymptotic stability in dynamic systems), while asymptotic stability focuses on the stability of a platoon of vehicles over space under the influence of a small perturbation that is originated from the first vehicle of the platoon (it is asymptotically stable if the perturbation strictly diminishes over space/vehicles). Some researchers in traffic flow modelling also call these two types of stability analysis as single vehicle stability (or platoon stability/plant stability), and

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stability over vehicles (or string stability), respectively. For more information on different types of stability, see Wilson and Ward (2011) and Treiber and Kesting (2013). In the rest of this paper, unless stated otherwise they are referred to as local stability and string stability, respectively.

Meanwhile, as a core component of traffic flow theories, numerous CF models have been developed in the literature to realistically describe longitudinal vehicular interactions from various perspectives. See (Brackstone and McDonald, 1999; Saifuzzaman and Zheng, 2014; Toledo, 2007) for reviews on CF models. For the convenience of discussion, these CF models are grouped into three categories in this paper: basic CF models (B-CF), time-delayed CF models (TD-CF), and multi-anticipative/cooperative CF models (MAC-CF), as briefly defined below.

- B-CF models: this type of CF models describes longitudinal vehicular interactions between two consecutive vehicles without considering any time delays.
- TD-CF models: compared with B-CF models, this type of CF models considers time delay related to CF.
- MAC-CF models: this type of CF models considers influence of more than one leading vehicle on CF behaviour with or without time delays.

Researchers started implementing stability analysis in CF modelling approximately 60 years ago. Generally, stability analysis in each category of CF models has different requirements and challenges. Although there has been significant progress with various methodologies existing for stability analysis of CF models, as the concept of stability is from the control theory and many methods for stability analysis are developed in other disciplines and usually mathematically heavy, it seems that the traffic flow community is, by and large, still not familiar with many of these methods. A general framework for stability analysis of B-CF has been proposed in Wilson and Ward (2011) to facilitate an easier implementation of stability analysis for researchers in traffic flow. However, a comprehensive review of CF stability analysis studies and a detailed description and comparison of major stability analysis methods are still missing, despite its great need. Particularly, CF model stability analysis' practical implications are largely ignored, and no studies have assessed the consistency between the theoretical criteria (from these stability analysis methods) and the simulation results, and in turn the applicability of the stability criteria obtained from various methods. Moreover, the advent of connected and autonomous vehicles (CAVs) can positively or negatively affect traffic flow's stability (Talebpour and Mahmassani, 2016), which is likely to bring numerous opportunities and challenges of utilising stability analysis and developing effective control strategies accordingly by taking advantage of the connected environment . This further underscores the importance and urgency of revisiting and comparing major stability analysis methods, and assessing their applicability to guide future research related to the stability of traffic flow consisting of traditional, connected, and autonomous vehicles.

This paper fills this gap. For the sake of clarity and focus, the paper concentrates on commonly-used methods for linear stability analysis of representative CF models, rather than attempting to be exhaustive. For each type of CF models, notable methods in the literature for analysing its local stability and string stability are reviewed in detail, including the characteristic equation based method (e.g., root extracting, the root locus method, the Routh-Hurwitz criterion, the Nyquist criterion and the Hopf bifurcation method), Lyapunov criterion, the direct transfer function based method, and the Laplace transform based method. In addition, consistency and applicability of stability criteria obtained using many of these methods are objectively compared with the simulation result from a series of numerical experiments. Particularly, the impact of connectivity and its interaction with time delay on stability is also investigated. Finally, issues, challenges, and research needs of CF models' stability analysis in the era of connected and autonomous vehicles are discussed.

The remainder of this paper is organized as follows: Section 2, 3, and 4 review and compare major stability analysis methods for B-CF, TD-CF, and MAC-CF models in the literature, respectively; Section 5 discusses strengths, issues, challenges, and research needs of CF models' stability analysis; and Section 6 summaries main findings of this study.

2. Stability analysis of B-CF models

This section reviews major methods for both local stability and string stability analysis of B-CF models.

2.1. B-CF models

For the convenience of discussion, a typical CF schematic is presented in Fig.1, where identical vehicles follow each other in a single lane without overtaking. The vehicles are labelled 1, 2, …, *n*-1, *n* in the upstream direction. The *n*th vehicle's position, speed and acceleration at time *t* is denoted as $x_n(t)$, $v_n(t)$ and $a_n(t)$, respectively. Spacing $\Delta x_n(t) = x_{n-1}(t) - x_n(t)$ (the front bumper of the leading vehicle to the front bumper of the following vehicle; gap $s_n(t) = \Delta x_n(t) - l_{veh}$ can be used instead, which is from the rear bumper of the leading vehicle to the front bumper of the following vehicle) and speed difference $\Delta v_n(t) = v_{n-1}(t) - v_n(t)$ are also essential variables for most CF models. Note that whether using spacing $\Delta x_n(t)$ or gap $s_n(t)$ generally has no influence on the analytical result since the leading vehicle's length l_{veh} is cancelled out during differentiation. Different notations are also used in the literature. For consistency and clarity, the notations depicted in Fig. 1 are used throughout this paper.

In general, the driver's response in a time-continuous model is directly represented by the acceleration function $a_n(t)$ in terms of the gap $s_n(t)$ and the speed difference $\Delta v_n(t)$ to the leading vehicle, and the driver's speed $v_n(t)$, and as a set of ordinary differential equations (ODEs):

$$\dot{x}_n(t) = v_n(t) \tag{1}$$

$$\dot{v}_n(t) = f(s_n, v_n, \Delta v_n)_t \tag{2}$$

Sometimes, the speed difference $\Delta v_n(t)$ is replaced by the direct use of the leading vehicle's speed $v_{n-1}(t)$ and then Eq. (2) becomes $\dot{v}_n(t) = f(s_n, v_n, v_{n-1})_t$. However, the analytical process for the two types of formulation is almost the same and their results can be derived from each other (Treiber and Kesting, 2013). For clarity, Eq. (2) is considered as the basic acceleration form in this paper.

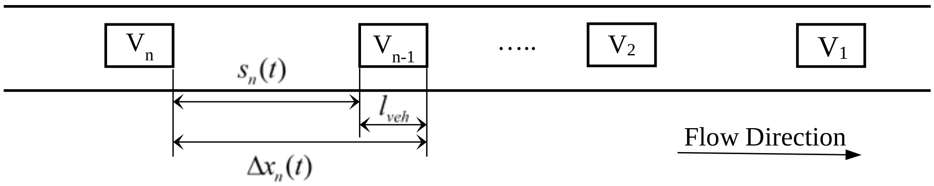


Fig. 1. A typical car following schematic.

As mentioned in the introduction, stability is considered as the ability of one vehicle (platoon) reacting to small perturbations with respect to the steady/equilibrium state where vehicles have identical gap s_e and speed v_e in homogenous traffic ($f(s_e, v_e, 0) = 0$). In CF models, small changes in gap, speed, or acceleration can be treated as sources of small perturbations. In the literature, gap variation $y_n(t)$ (see Eq. (3)) and speed variation $u_n(t)$ (see Eq. (4)) with respect to the equilibrium state are frequently used.

$$y_n(t) = s_n(t) - s_e \tag{3}$$

$$u_n(t) = v_n(t) - v_e \tag{4}$$

Therefore, the analysis of stability is to evaluate the evolution of $y_n(t)$ or $u_n(t)$ with respect to time *t* (local stability) or vehicle *n* (string stability).

2.2. Local stability analysis

Although local stability analysis of CF models has been conducted since 1959 (Herman et al., 1959), due to its relative simplicity compared with string stability analysis, it did not attract as much attention as string stability analysis in the literature. However, it is an important feature that a CF model should have because typically a driver is able to easily recover from small disturbances in real traffic and return to a steady CF state over time. Thus, local stability is still worth investigating, especially for CF models that are newly developed.

For the local stability analysis of B-CF models, there exists three major methods. The first method is the characteristic equation based method, which is on the basis of the solution of characteristic equation (an algebraic equation which depends on the solution of a given differential equation) of ODEs (Aström and Murray, 2010; Olver and Shakiban, 2007). The second method is the Laplace transform based method, which combines the Laplace transform and the characteristic equation based method together (Herman et al., 1959). Although the characteristic equation is also used in this method, the way of obtaining the characteristic equation is different, and more importantly, the Laplace transform based method can also be used in the string stability analysis in a different way, as discussed later. Thus, the Laplace transform based method is treated as a different method in this paper. The third method is Lyapunov's second method (which is often referred to as the Lyapunov stability criterion) proposed by Lyapunov in his original 1892 work (Lyapunov, 1992).

2.2.1. The characteristic equation based method

The characteristic equation based method aims to assess the perturbation's growth rate from the solution of the perturbation's mathematical form, which is usually in forms of nonlinear ODEs. For the simplest linear ODE: $\frac{dx}{dt} = ax$, we can easily get the solution with the knowledge of elementary calculus: $x = x_0 e^{at}$. Then for the solution of the first-order homogenous linear system of ODEs (i.e., ODEs that do not contain a constant term and in which the unknowns only have the first power (Kreyszig, 2010)), which can be written in vector form: $\dot{\mathrm{x}}(\mathrm{t}) = \frac{d\mathrm{x}(\mathrm{t})}{dt} = A\mathrm{x}(\mathrm{t})$, where $A \in \mathbb{R}^{n\times n}$ is a square matrix, $\mathrm{x}(t) = [x_1, x_1, ..., x_n]$ is a set of state variables and its original equilibrium vector is $\dot{\mathrm{x}}(0)=0$, we can make an ansatz (i.e., the method of finding a complicated equation's solution by guessing the solution's form in advance (Gershenfeld, 1999)): $\mathrm{x}(t) = e^{\lambda t}\mathrm{v}$. The exponential function $\mathrm{x}(t) = e^{\lambda t}\mathrm{v}$ is a (non-zero) solution if and only if λ is an eigenvalue of A and v is the corresponding eigenvector. In addition, according to the superposition principle of the homogenous system, the general solution of the first-order homogenous linear ODEs system is $\mathrm{x}(t) = c_1\mathrm{x}_1(t) + \cdots c_n\mathrm{x}_n(t) = c_1 e^{\lambda_1 t}\mathrm{v}_1 + \cdots c_n e^{\lambda_n t}\mathrm{v}_n$ (Aström and Murray, 2010). The system is stable when $\mathrm{x}(t)$ approaches to zero over time. More specifically, the system's stability is determined by λ. If λ is real ,it should be negative; if λ is complex, its real part should be negative, because the imaginary part of the eigenvalue only contributes to the oscillatory component of the solution (Strang, 1991). Thus, a general form of the stability criterion can be expressed as $\mathrm{Re}\lambda < 0$ .As most CF models are nonlinear, they are often linearized through the first-order Taylor expansion before the stability criterion above is applied. Suppose that we have a set of nonlinear ODEs in a vector form: $\dot{\mathrm{x}}(t) = \mathrm{F}(\mathrm{x})$ where $\mathrm{x}(t) = [x_1, x_1, ..., x_n]$ is a state vector and $\mathrm{F} = [f_1, f_1, ..., f_n]$ is a function vector in which each function is continuously differentiable at least once. x_e is an equilibrium vector of the system, i.e., $\dot{\mathrm{x}}_e = \mathrm{F}(\mathrm{x}_e) = 0$. For the linear stability analysis with small perturbations, we can take a multivariate first order Taylor expansion of the right-hand side around the equilibrium point (linearization), as shown in Eq. (5).

$$\dot{\mathrm{x}}(t) \approx \mathrm{F}(\mathrm{x_e}) + \left.\frac{\partial f}{\partial \mathrm{x}}\right|_{\mathrm{x}_e} (\mathrm{x} - \mathrm{x_e}) = \left.\frac{\partial f}{\partial \mathrm{x}}\right|_{\mathrm{x}_e} (\mathrm{x} - \mathrm{x_e}) \tag{5}$$

Here, $\left.\frac{\partial f}{\partial \mathrm{x}}\right|_{\mathrm{x}_e} = \mathrm{J}_e$ denotes the $n \times n$ Jacobian matrix J of the partial derivative at the equilibrium point.

The deviation from the equilibrium, $\delta\mathrm{x} = \mathrm{x} - \mathrm{x}_e$, is governed by a linearized system: $\delta\dot{\mathrm{x}} = \dot{\mathrm{x}} = \mathrm{J}_e \delta\mathrm{x}$. According to the stability criterion discussed above for the linear system, the nonlinear system around the equilibrium point x_e is stable if and only if all the eigenvalues of the coefficient matrix J_e have a negative real part.

Regarding CF models shown in Eq. (1) and (2), we first linearize Eq. (2):

$$\dot{v}_n(t) \approx f(s_e, v_e, 0)_t + \left.\frac{\partial f}{\partial s}\right|_e (s_n(t) - s_e) + \left.\frac{\partial f}{\partial v}\right|_e (v_n(t) - v_e) + \left.\frac{\partial f}{\partial \Delta v}\right|_e (\Delta v_n(t) - 0) \tag{6}$$

With $f(s_e, v_e, 0)_t = 0$, Eq. (3) and (4) and $\Delta v_n = u_{n-1} - u_n$, Eq. (6) can be transformed to

$$\dot{v}_n(t) \approx (f_s y_n + f_v u_n + f_{\Delta v}(u_{n-1} - u_n))_t \tag{7}$$

where $f_s = \left.\frac{\partial f}{\partial s}\right|_e$, $f_v = \left.\frac{\partial f}{\partial v}\right|_e$, $f_{\Delta v} = \left.\frac{\partial f}{\partial \Delta v}\right|_e$ are the Taylor expansion coefficients.

An ODE system of deviations from the equilibrium, i.e., the gap variation $y_n(t)$ and the speed variation $u_n(t)$, thus can be derived as shown in Eq. (8) and (9).

$$\dot{y}_n(t) = \dot{s}_n(t) = \dot{x}_{n-1}(t) - \dot{x}_n(t) = v_{n-1}(t) - v_n(t) = u_{n-1}(t) - u_n(t) \tag{8}$$

$$\dot{u}_n(t) = \dot{v}_n(t) = (f_s y_n + (f_v - f_{\Delta v})u_n + f_{\Delta v}u_{n-1})_t \tag{9}$$

For the local stability, the leading vehicle could be in the equilibrium state while there is an initial perturbation of the follower, that is, $u_{n-1}(t) = 0$. Thus, Eq. (8) and (9) can be further simplified as the linear ODE system defined in Eq. (10) and (11).

$$\dot{y}_n(t) = -u_n(t) \tag{10}$$

$$\dot{u}_n(t) = (f_s y_n + (f_v - f_{\Delta v})u_n)_t \tag{11}$$

Using the results previously discussed, the CF model is locally stable if and only if the eigenvalues of the coefficient matrix $\mathrm{J}_e = \begin{bmatrix} 0 & -1 \\ f_s & f_v - f_{\Delta v} \end{bmatrix}$ have negative real parts. The eigenvalues λ can be calculated by $|(\mathrm{J}_e - \lambda I)| = 0$ ($I = diag(1,1,\ldots,1)$ is the identity matrix with a correct size), which is referred to as the characteristic equation:

$$\lambda^2 - (f_v - f_{\Delta v})\lambda + f_s = 0 \tag{12}$$

with the solutions:

$$\lambda_{\pm} = \frac{(f_v - f_{\Delta v}) \pm \sqrt{(f_v - f_{\Delta v})^2 - 4f_s}}{2} \tag{13}$$

The CF model is locally stable when the real parts of the two solutions are both negative.

An alternative way to obtain the characteristic equation is to assume the exponential ansatz $y_n(t) = y_n^0 e^{\lambda t}$ and $u_n(t) = u_n^0 e^{\lambda t}$, where y_n^0, u_n^0 are initial perturbations, and then insert them into Eq. (10) and (11) directly. This approach for local stability was adopted by Treiber and Kesting (2013). A summary of representative studies of stability analysis of CF models is presented in Table A1 in Appendix.

Note that the partial derivatives of reasonable CF models should satisfy some rational driving constraints defined in Eq. (14) (Wilson and Ward, 2011), because in spite of other factors, a larger spacing should result in more acceleration (or less braking); a larger velocity should make a driver less likely to accelerate (or more likely to brake); and a larger (more positive) relative velocity should result in more acceleration (or less braking).

$$f_s > 0,\ f_v < 0 \text{ and } f_{\Delta v} > 0. \tag{14}$$

For reasonable CF models that satisfies these rational driving constraints, both the solutions of Eq. (12) have negative real parts, which means that the local stability of such CF models is automatically guaranteed.

Additionally, an indirect mathematical technique can deal with the quadratic characteristic equation such as Eq. (12), as explained below.

Routh-Hurwitz stability criterion: Given a polynomial: $f(z) = z^2 + a_1 z + a_2$ where $a_1, a_2 \in \mathbb{R}$. $f(z)$ is stable if and only if $a_1 > 0, a_2 > 0$.(Ogata, 2010)

According to the Routh-Hurwitz stability criterion, the local stability criterion of B-CF models is

$$(f_v - f_{\Delta v}) < 0 \text{ and } f_s > 0 \tag{15}$$

It is easy to verify that this criterion can also be automatically satisfied with the rational driving constraints. When a_1 and a_2 are not real, by separating the real part and imaginary part of the characteristic equation and obtaining two sub-equations with real parameters, the Routh-Hurwitz criterion can then be applied.

2.2.2. The Laplace transform based method

Another approach to obtain the characteristic equation is to take the Laplace transform. $U(s)=\mathcal{L}(u(t))$, $Y(s)=\mathcal{L}(y(t))$ denote the Laplace transform of speed variation $u(t)$ and spacing variation $y(t)$, respectively, where $\mathcal{L}$ is the Laplace transform function. Then Eq. (8) and (9) can be rearranged as:

$$Y(s)=\mathcal{L}(\int(u_{n-1}-u_n)dt)=\frac{1}{s}[\mathcal{L}(u_{n-1}(t))-\mathcal{L}(u_n(t))]=\frac{1}{s}[U_{n-1}(s)-U_n(s)] \tag{16}$$

$$sU_n(s)=\frac{1}{s}f_s[U_{n-1}(s)-U_n(s)]+f_vU_n(s)+f_{\Delta v}(U_{n-1}(s)-U_n(s)) \tag{17}$$

We can obtain the transfer relationship between the leader and the follower in the Laplace frequency domain:

$$U_n(s)=\frac{sf_{\Delta v}+f_s}{s^2-s(f_v-f_{\Delta v})+f_s}U_{n-1}(s) \tag{18}$$

The speed variation of the second vehicle in the time domain is thus obtained by taking the inverse Laplace transform:

$$u_2(t)=\mathcal{L}^{-1}\left[\frac{sf_{\Delta v}+f_s}{s^2-s(f_v-f_{\Delta v})+f_s}U_1(s)\right] \tag{19}$$

According to the characteristics of the inverse Laplace transform, the poles[1] s_p of the right-hand side contribute terms of type $e^{s_p t}$ to $u_2(t)$ when the transfer function is transformed into the time domain, and turn the denominator to the denominator $s^2-s(f_v-f_{\Delta v})+f_s=0$ the characteristic equation (Wilson and Ward, 2011). This characteristic equation is identical with that derived from the characteristic equation based method (i.e., Eq. (12)). Thus, the stability criterion from the Laplace transform based method is also the same as that from the characteristic equation based method. The Laplace transform based method was utilized by Wilson and Ward (2011) and Li et al. (2013) to analyse a B-CF model's local stability.

2.2.3. Lyapunov stability criterion

The characteristic equation based method discussed above needs to examine whether the solutions have negative real parts, which sometimes is difficult to do because of the complexity of computing the eigenvalues or applying the Routh-Hurwitz stability criterion. Thus, Lyapunov's second method (the Lyapunov stability criterion hereafter) is a useful alternative for analysing the stability of a linear or nonlinear system because this method does not need to compute the eigenvalues of the system(LaSalle and Lefschetz, 1961).

The central idea of the Lyapunov stability criterion is to make use of a Lyapunov function $V(x)$, which has an analogy to the potential energy function of classical dynamics of particles and systems. If we can find a non-negative energy-like function (energy cannot be negative) that always decreases along trajectories of the system, a locally stable equilibrium point can be identified simply as the minimum of the function.

For a system $\dot{x}(t)=\mathrm{F}(\mathrm{x})$ with an equilibrium point at x_e, a function $V(x)$ is called a Lyapunov function candidate if the following conditions are satisfied: i) $V(\mathrm{x})=V_0$ if and only if $\mathrm{x}=\mathrm{x}_e$; ii) $V(\mathrm{x})>V_0$ if and only if $\mathrm{x}\neq\mathrm{x}_e$ (positive definite); and iii) $\dot{V}(\mathrm{x})=\frac{\partial V}{\partial \mathrm{x}}\frac{d\mathrm{x}}{dt}=\frac{\partial V}{\partial \mathrm{x}}\mathrm{F}(\mathrm{x})\leq 0$ for all values of $\mathrm{x}\neq\mathrm{x}_e$ (negative semidefinite).

If such a Lyapunov function exists, the system is stable in the sense of Lyapunov. For locally asymptotic stability (local stability in CF modelling), $\dot{V}(x)<0$ is required (negative definite).

[1] Note that pole is a commonly-used concept in complex analysis and also appears many times later in this paper. Without delving into a strict mathematical definition, a pole is similar to a singular point, and at a pole, a function approaches infinity.

Obviously, the critical step of using the Lyapunov stability criterion is to confirm the existence of a Lyapunov function. It has been proved that for a two-state-variable ODEs system: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $a,b,c,d \in \mathbb{R}$, x_1, x_2 is the state variable, if $a \le 0,\ d \le 0,\ bc < 0$, there exists a Lyapunov function $V(x) = px_1^2 + qx_2^2 \left(p,q \in \mathbb{R},\ p > 0,\ q > 0\right)$. (Li et al., 2010)

Then for the local stability analysis of CF models in forms of Eq. (10) and (11), we rewrite them as:

$$\begin{bmatrix} \dot{y} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ f_s & f_v - f_{\Delta v} \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{20}$$

When $f_s > 0$ and $f_v - f_{\Delta v} < 0$, a Lyapunov function that ensures the local stability is

$$\begin{aligned} &V(x) = f_s y^2 + u^2 > 0 \text{ when } y,u \ne 0 \\ &\dot{V}(x) = 2(f_v - f_{\Delta v})u^2 < 0 \end{aligned} \tag{21}$$

This criterion is the same with that obtained using the characteristic equation based method or using the Laplace transform based method. the Lyapunov stability criterion was applied to the full velocity difference model in Li et al. (2010).

2.3. String stability analysis

Unlike local stability, to be string stable requires that the perturbation strictly attenuates for each leader-follower pair as it propagates away from the first leader (Liu et al., 2001; Treiber and Kesting, 2013), as defined in Eq. (22):

$$\left\|\varepsilon_1\right\|_\infty > \left\|\varepsilon_2\right\|_\infty > \left\|\varepsilon_3\right\|_\infty > \cdots > \left\|\varepsilon_n\right\|_\infty \tag{22}$$

where $\left\|\varepsilon_n\right\|_\infty = \max_t \left|\varepsilon_n(t)\right|$ is the maximum magnitude of the perturbation within infinite time. Note that this definition is consistent with the definition of string stability in Treiber and Kesting (2013), and if any inequalities in Eq. (22) do not hold, the platoon is regarded as being unstable.

The string stability analysis is particularly relevant to traffic flow modelling as the stop-and-go oscillation in real traffic is a manifestation of the string instability of traffic flow. From the CF model development perspective, for a realistic CF model, it should be able to depict the string instability in order to replicate characteristics related to traffic oscillations; however, from traffic operation perspective, string instability should be minimized or avoided. Thus, the string stability analysis has been frequently implemented since the early-stage of CF model development (Chandler et al., 1958), and several methods have been proposed: the direct transfer function based method, the Laplace transform based method, the characteristic equation based method (including root extracting and the root locus method). Detail of each method is provided below.

2.3.1. The direct transfer function based method

The basic idea of the direct transfer function based method is to look at the frequency response between the input and output of a system. By viewing propagation of the perturbation between two consecutive vehicles as a system (Aström and Murray, 2010), it can be naturally applied to the string stability analysis of CF models.

According to Fourier theory that any periodic signal can be decomposed into a sum of pure periodic sine and cosine waves, an input signal of a system that is periodic can be written as (Olver and Shakiban, 2007):

$$h(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega t) + b_k \cos(k\omega t) \tag{23}$$

where ω is the fundamental frequency of the periodic input, a_k, b_k, k are coefficients. Each of the terms in this input generates a corresponding trigonometric output (in a steady state), with a possibly shifted magnitude and phase. The magnitude gain and shifted phase are determined by the frequency response $G(s)$, which is referred to as the transfer function. By computing the gain of magnitude, the relationship between input and output is obtained. Thus, variation of the perturbation of two vehicles (then a platoon) can be evaluated to assess the string stability. This method is called the direct transfer function based method in this paper because of the central role of transfer function, although transfer function also appears in other methods, e.g., the Laplace transform based method and the Nyquist stability criterion.

Because of the common use of exponential forms and the simple transformation between exponential signals and trigonometric signals: $e^{\lambda t} = e^{(\sigma+i\omega)t} = e^{\sigma t}(\cos\omega t + i\sin\omega t)$, $h(t) = e^{st}$ is used as the input of a linear input/output system described by the general controlled differential equation:

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n x = b_0 \frac{d^m h}{dt^m} + b_1 \frac{d^{m-1}h}{dt^{m-1}} + \cdots + b_m h \tag{24}$$

where the output of the system is $x(t) = x_0 e^{st}$; $a_1, \cdots, a_n$ and $b_1, \cdots, b_m$ are coefficients. With the substitution of input and output, Eq. (24) is rewritten as:

$$(s^n + a_1 s^{n-1} + \cdots a_n) x_0 e^{st} = (b_0 s^m + b_1 s^{m-1} + \cdots b_m) e^{st} \tag{25}$$

Then the output can be re-organised as:

$$x(t) = x_0 e^{st} = \frac{b(s)}{a(s)} h(t) \tag{26}$$

where $a(s) = s^n + a_1 s^{n-1} + \cdots a_n$, and $b(s) = b_0 s^m + b_1 s^{m-1} + \cdots b_m$. The transfer function between the input and the output is $G(s) = b(s)/a(s)$.

Further by setting $s = i\omega$ (this is commonly used in control theory to initialise a steady signal in the frequency domain and also frequently used in the following sections), which ensures the stability of the input signal:

$$x(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\varphi)} = M\cos(\omega t+\varphi) + iM\sin(\omega t+\varphi) \tag{27}$$

Here, $M = |G(i\omega)|$ and $\varphi = \arctan\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}$ are the magnitude gain and shifted phase of $G(s)$, respectively. $M = |G(i\omega)| < 1$ leads to $\|x(t)\| < \|h(t)\|$, which keeps the system string stable.

For B-CF models, we assume that the perturbation of the leading vehicle is steady oscillation $u_0(t) = e^{i\omega t}$. Then $u_n(t) = G^n(s)e^{i\omega t}$. By substituting it into Eq. (8) and (9) we have:

$$G(i\omega) = \frac{f_s + i\omega f_{\Delta v}}{-\omega^2 - i\omega(f_v - f_{\Delta v}) + f_s} \tag{28}$$

So that,

$$u_n(t) = |G(i\omega)|^n e^{i(\omega t + n\varphi)} \tag{29}$$

It is easy to show that the perturbation in a platoon of traffic will damp out rather than amplify if:

$$|G(i\omega)| = \sqrt{\omega^2 f_{\Delta v}^2 + f_s^2} \Big/ \sqrt{(f_s - \omega^2)^2 + \omega^2 (f_v - f_{\Delta v})^2} < 1 \tag{30}$$

Thus, the string stability criterion is

$$\omega^2 f_{\Delta v}^2 + f_s^2 < (f_s - \omega^2)^2 + \omega^2 (f_v - f_{\Delta v})^2 \tag{31}$$

Since low frequency $\omega \to 0$ gives the strongest constraint on stability (long-wavelength instability always occurs first), the following inequality has to be satisfied to ensure the string stability:

$$f_v^2 - 2f_s - 2f_v f_{\Delta v} > 0 > -\omega^2 \text{ or } \frac{1}{2} - \frac{f_{\Delta v}}{f_v} - \frac{f_s}{f_v^2} > 0 \tag{32}$$

The direct transfer function based method was first implemented in CF modelling by Chandler et al. (1958) with a relative speed control model.

2.3.2. The Laplace transform based method

Similar to the direct transfer function based method, Laplace transform can facilitate the string stability analysis of a system by transforming the system from the time domain to the frequency domain. The relationship of the perturbation between two consecutive vehicles in the frequency domain is:

$$G(s) = E_n(s) / E_{n-1}(s) \tag{33}$$

where $E_n(s)$ is the Laplace transform of $\varepsilon_n(t)$. Let $g(t)$ be the inverse Laplace transform of $G(s)$, we have:

$$\begin{aligned} \|\varepsilon_n\|_\infty &= \max_t |\varepsilon_n(t)| = \max_t \left|\int_0^t g(\tau)\varepsilon_{n-1}(t-\tau)d\tau\right| \le \max_t \int_0^\infty |g(\tau)||\varepsilon_{n-1}(t-\tau)|d\tau \\ &\le \int_0^\infty |g(\tau)|d\tau \max_t |\varepsilon_{n-1}(t-\tau)| = \|g\|_1 \|\varepsilon_{n-1}\|_\infty \end{aligned} \tag{34}$$

where $\|g\|_1 = \int_0^\infty |g(\tau)|d\tau$. Obviously, to satisfy the string stability, $\|g_1\| < 1$.

Meanwhile, if $s = i\omega$, according to the definition of Laplace transform, we have:

$$|G(i\omega)| = \left|\int_0^\infty g(t)e^{-i\omega t}dt\right| \le \int_0^\infty |g(t)||e^{-i\omega t}|dt = \int_0^\infty |g(t)|dt = \|g\|_1 \tag{35}$$

Therefore, one sufficient condition for string stability of the system is

$$|G(i\omega)| \le \|g\|_1 < 1 \tag{36}$$

As shown in Eq. (18), for B-CF models, the transfer relationship between the leader and the follower in the frequency domain is

$$G(s) = \frac{U_n(s)}{U_{n-1}(s)} = \frac{sf_{\Delta v} + f_s}{s^2 - s(f_v - f_{\Delta v}) + f_s} \tag{37}$$

By inserting $s = i\omega$ into Eq. (37), it becomes identical with Eq. (28). Thus, the same string stability criterion obtained from using the direct transfer function based method can be derived by using Laplace transform (Liu et al., 2001).

2.3.3. The characteristic equation based method

As discussed in Section 2.2.1, the characteristic equation can be obtained with a proper ansatz of perturbation. The perturbation of the leading vehicle cannot be set to zero when studying the propagation of perturbations along a platoon. Therefore, unlike in local stability analysis, $y_n(t) = y_n^0 e^{\lambda t}$ and $u_n(t) = u_n^0 e^{\lambda t}$ (see Section 2.2.1) cannot be directly used for string stability. However, inspired by Eq. (29), we can represent $u_n(t)$ and $y_n(t)$ as in Eq. (38) and (39), respectively.

$$u_n(t) = u_n^0 e^{\lambda t} = \hat{u} e^{in\varphi} e^{\lambda t} = e^{\lambda t + in\varphi}\hat{u} \tag{38}$$

$$y_n = e^{\lambda t + in\varphi}\hat{y} \tag{39}$$

where $\lambda = \sigma + i\omega$ is the complex growth rate; the real part σ denotes the growth rate of the oscillation magnitude while the imaginary part ω indicates the frequency, $\hat{u}, \hat{y}$ are complex constants (independent of *n* and *t*). The driver passes a complete wave in the time $2\pi / \omega$. Shifted phase (or wavenumber) $\varphi \in [-\pi, \pi]$ indicates the phase shift of the traffic waves from one vehicle to the next at a given time, and the number of vehicles per wave is $2\pi / \varphi$ (Treiber and Kesting, 2013).

Inserting Eq. (38) and (39) into Eq. (8) and (9) gives us Eq. (40).

$$\begin{pmatrix} \lambda & (1 - e^{-i\varphi}) \\ -f_s & \lambda - (f_v - f_{\Delta v} + f_{\Delta v} e^{-i\varphi}) \end{pmatrix} \begin{pmatrix} \hat{y} \\ \hat{u} \end{pmatrix} = 0 \tag{40}$$

This homogeneous two-variable linear system has non-zero solutions only if the determinant of the square matrix of coefficients is singular (Olver and Shakiban, 2007), i.e., its determinant is equal to zero, as shown in Eq. (41).

$$\lambda^2 + p(\varphi)\lambda + q(\varphi) = 0 \quad \left(p(\varphi) = -f_v + f_{\Delta v} - f_{\Delta v}e^{-i\varphi}; q(\varphi) = f_s(1-e^{-i\varphi})\right) \tag{41}$$

Solving Eq. (41) gives:

$$\lambda_{\pm} = -\frac{p(\varphi)}{2}\left(1 \pm \sqrt{1 - \frac{4q(\varphi)}{p^2(\varphi)}}\right) \tag{42}$$

Obviously, if the real part of λ_+ is negative for all φ, the CF model is string stable. Moreover, the first instability of B-CF models always occurs when phase shift $\varphi \to 0$, as waves with a finite wavelength $2\pi / \varphi$ can only have finite growth rates. We expand λ_+ around $\varphi \to 0$:

$$\lambda_+ = i\frac{f_s}{f_v}\varphi + \frac{f_s}{f_v}\left(\frac{1}{2} - \frac{f_{\Delta v}}{f_v} - \frac{f_s}{f_v^2}\right)\varphi^2 + \ldots \tag{43}$$

To satisfy the stability criterion, the real part of λ_+ should be negative. Since $f_s > 0$, $f_v < 0$ and $f_{\Delta v} > 0$ for rational driving behaviours, the stability criterion identical with that from using the direct transfer function based method (see Eq. (32)) is obtained.

Another way to obtain the characteristic equation is to consider a ring road configuration, where N ($N \to \infty$) vehicles are evenly placed. By assuming $u_n(t) = u_n^0 e^{\lambda t}$ and inserting it into Eq. (8) and (9), the *N*th order characteristic equation is obtained as shown in Eq. (44), in which the (*N+1)*th vehicle is just the *1*st vehicle:

$$(\lambda^2 - (f_v - f_{\Delta v})\lambda + f_s)^N = (f_s + f_{\Delta v}\lambda)^N \tag{44}$$

Taking the *N*th root of Eq. (44) leads to an equation identical to Eq. (41). Thus, the remaining steps for deriving the stability criterion are the same, and not repeated here. Orosz et al. (2010b) applied the characteristic equation based method and the Laplace transform based method to a basic CF model, and obtained the same conclusion.

Besides directly extracting the roots of the characteristic equation, substituting $\lambda = i\omega$ (when the real part of λ is 0, the system stays at the boundary between stability and instability) and $e^{i\varphi} = \cos\varphi + i\sin\varphi$ into the characteristic equation, and then separating the real part and imaginary part leads to Eq. (45) and (46) that describe the stability changes (Hopf bifurcations):

$$f_s = \frac{1}{2}(2f_{\Delta v} - f_v)\left[(2f_{\Delta v} - f_v)\tan^2(\frac{\varphi}{2}) - f_v\right] \tag{45}$$

$$\omega = (2f_s - f_v)\tan(\frac{\varphi}{2}) \tag{46}$$

As mentioned before, the instability always first occurs at $\omega \to 0$. The stability criterion identical to Eq. (32) can be obtained. This Hopf bifurcation method was adopted in Orosz et al. (2011); Orosz et al. (2010b).

2.3.4. The root locus method

Based on the result of the characteristic equation based method, a further step can be taken to examine how variation of a certain system parameter changes the roots of a system (consequently the stability). The root locus method is a simple but powerful graphical tool developed by Evans (1948) for this purpose. Note that in mathematics, a locus is a set of points (a curve in our case), whose location is determined by particular mathematical equations (James and James, 1992).

For the characteristic equation shown in Eq. (41), it can be further simplified by setting $s = -\lambda / f_v$, and the roots of the resulting equation are

$$s_{\pm} = \frac{1}{2}(aK - 1 \pm \sqrt{(aK-1)^2 + 4bK}) \tag{47}$$

where $a = -\frac{f_{\Delta v}}{f_v}, b = \frac{f_s}{f_v^2}$, and $K = (e^{-i\varphi} - 1)$.

Then the root locations in the complex plane with different φ can be drawn accurately. If all the roots (except the root at $\varphi = 0$) locates at the left half plane (LHP), which implies a negative real part, the stability criterion is satisfied. Sau et al. (2014) used the root locus method to show string stability conditions of basic and some cooperative CF models.

2.4. Numerical simulation

2.4.1 Simulation design

Although researchers started implementing the stability analysis of various CF models since the early days of traffic modelling, CF model stability analysis' practical implications are largely ignored. One of the main reasons is theoretical analysis of stability includes many assumptions which are not realistic in real traffic, such as arbitrarily long platoon, long-wavelength instability, steady oscillation, and etc. To use stability analysis to guide the development of traffic control strategies, the applicability of the stability analysis result needs to be sufficiently high. Meanwhile, an easy way of testing the stability of a vehicular platoon is to simply simulate a group of vehicles that moves along a single-lane straight road and encounters a disturbance in the first vehicle, and then decide the stability of this group of vehicle based on how the disturbance evolves over time along the first vehicle (i.e., local stability) or across the vehicles upstream (i.e., string stability). This method has been employed in several studies in the literature (Bando et al., 1998; Ferrari, 1994; Herman et al., 1959; Lenz et al., 1999; Monteil et al., 2014; Treiber and Kesting, 2013). Thus, a natural idea of assessing the consistency and applicability of the theoretical results of stability analysis in the literature is to compare theoretical results with simulation results. Checking important factors' impact on the consistency between the theoretical result of stability analysis and the simulation result can shed light on the value of stability analysis in practice, e.g., if the consistency is sensitive towards the platoon size, that would imply that stability analysis has limited value in practice because the platoon size in real traffic naturally varies significantly. Towards this end, a series of numerical simulations is designed, as shown in Fig. 2. For each category of CF models discussed in this paper, several main factors that may affect the comparison results are considered. Among these factors, characteristics of the disturbance appear in all three CF categories. More specifically, in our numerical simulations, intensity of the disturbance (measured by deceleration, and the range considered is 0.1-3 m/s^2 with a step size of $0.5m/s^2$), duration of the disturbance (measured by deceleration time, and the range considered is 0.1-6s with a step size of 0.5s), and magnitude of the disturbance (measured by the product of the intensity and duration, i.e., the total amount of the speed decrease; and the maximum magnitude of the disturbance considered is 15m/s, which is equal to the considered equilibrium speed). Each category of CF models also has its own unique factors that may influence its stability. For B-CF models, besides disturbance, the number of vehicles simulated is another factor in the numerical analysis, because in the real world, the length of a platoon cannot be infinity.

Either Spacing/speed deviation from the equilibrium spacing/speed or acceleration rate can be used as an indicator of disturbance. We adopt speed deviation in this paper for the sake of simplicity. As per the definition, if the absolute speed deviation from the equilibrium speed of the first following vehicle decrease over time, it is locally stable; otherwise, it is locally unstable; if the absolute speed deviation from the equilibrium speed of following vehicles strictly decrease along the platoon, the platoon is string stable; otherwise, it is string unstable. Since most CF models can easily satisfy the local stability criterion, the designed simulations are mainly for string stability analysis. The consistency of theoretical result and simulation result for string stability is evaluated using three indicators: overall consistency, stability consistency, and instability consistency as defined in Eq. (48). For the confusion matrix in Table 1, TP stands for the number of events in which theoretical stable points are identical with simulated stable points, FN for the number of events in which simulated stable points are identified as theoretical unstable points, FP for the number of events in which simulated unstable point are identified as theoretical stable points, and TN for the number of events in which simulated unstable point are identical with theoretical unstable points.

Table 1. Confusion matrix

	Theoretical stable point	Theoretical unstable point
Simulated stable point	True positive (TP)	False negative (FN)
Simulated unstable point	False positive (FP)	True negative (TN)

$$Overall\ consistency = (TP + TN)/(TP + FP + FN + TN)$$
$$Stablilty \text{ consistency} = TP/(TP + FN);\ Instablilty \text{ consistency} = TN/(TN + FP) \tag{48}$$

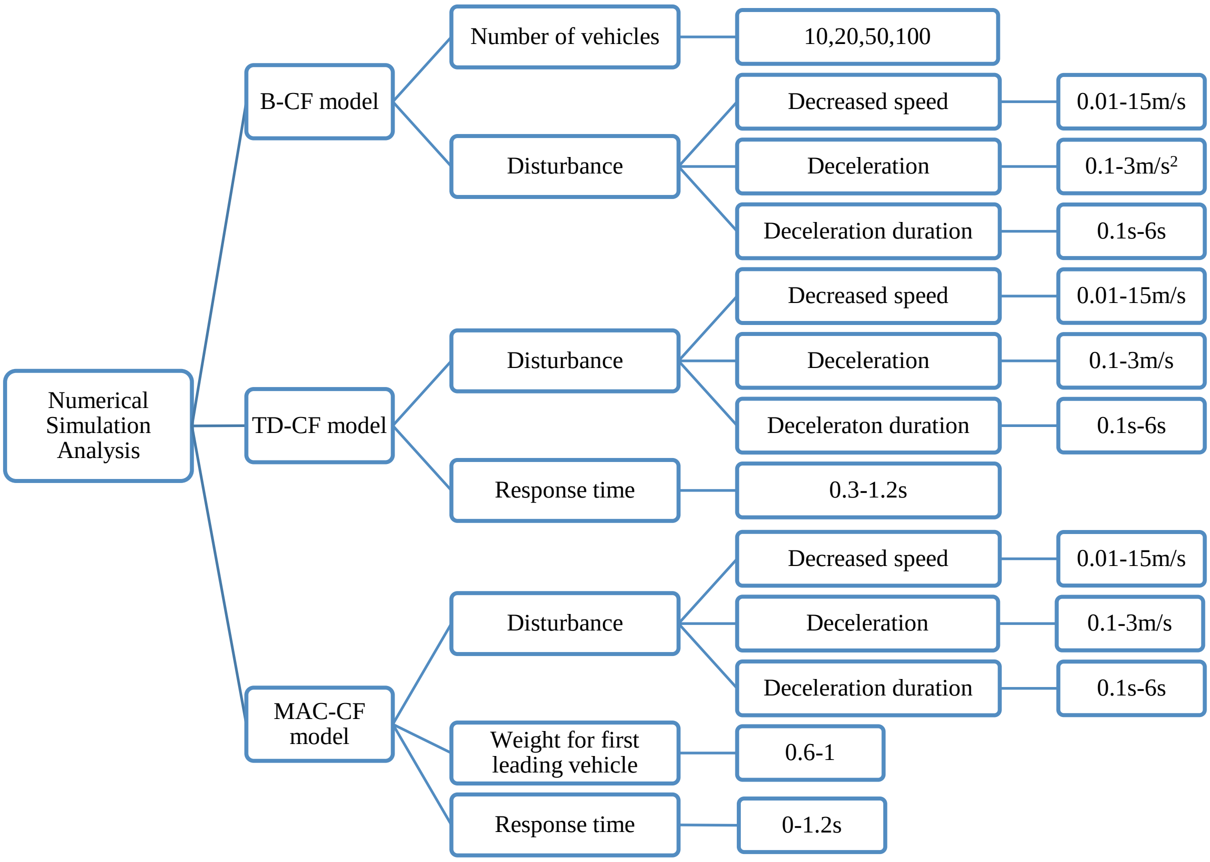


Fig. 2. Structure of numerical simulation analysis

The setup of the simulation is as follows: a platoon of identical vehicles enter a very long single-lane straight road, travelling at the same speed (15 m/s, i.e. 54km/h) with an equilibrium spacing (varies with different model parameters). Vehicles' speeds and positions are governed by a CF model. The intelligent driver model (IDM) developed by Treiber et al. (2000) is used as a B-CF model in this study mainly for three reasons: i) IDM is widely used in the literature, and its capability of satisfactorily reproducing many characteristics of traffic flow has been consistently demonstrated; ii) stability of IDM has been extensively analysed in the literature; and iii) IDM can be easily extended into the other two types of CF models: TD-CF, MAC-CF. However, using IDM in this study is only for demonstration purpose, and implementing similar numerical analysis using other CF models is straightforward.

IDM is mathematically formulated as in Eq. (49) and (50).

$$\dot{v}_n(t) = \alpha\left[1 - \left(\frac{v_n(t)}{v_0}\right)^4 - \left(\frac{s_n^*(t)}{s_n(t)}\right)^2\right] \tag{49}$$

$$s_n^*(t) = s_0 + Tv_n(t) - \frac{v_n(t)\Delta v_n(t)}{2\sqrt{\alpha\beta}} \tag{50}$$

where v_0 is the desired speed, $s_n^*(t)$ is the desired gap, s_0 is the minimum gap at the standstill situation, T is the desired time gap, α is the maximum acceleration and β is the comfortable deceleration. Other variables have the same meanings as those introduced in the general CF model.

The default parameter values used in this study are: v_0=120km/h; s_0=2m; β=1.5m/s^2; l_{veh} =5m. $a_n(t+\Delta T) = \dot{v}_n(t)$ is first calculated according to Eq. (49) and (50). The speed is then updated using the trapezoidal rule, as shown in Eq. (51), and the position is calculated with the kinematic equation as shown in Eq. (52) (Treiber and Kanagaraj, 2015):

$$v_n(t+\Delta T) = v_n(t) + \frac{1}{2}(a_n(t) + a_n(t+\Delta T))\Delta T \tag{51}$$

$$x_n(t+\Delta T) = x_n(t) + v_n(t)\Delta T + \frac{a_n(t)(\Delta T)^2}{2} \tag{52}$$

where ΔT is the updating time step and set as 0.1s in the simulations.

In the simulation, vehicles quickly reach the equilibrium state and then travel with the same speed and the corresponding equilibrium gap. Then at t=60s, a perturbation is introduced to the first leading vehicle by forcing it to first decelerate and then accelerate for the same time duration (i.e. the deceleration duration considered in Fig. 2). The speeds of the following vehicles from 50s to1000s, which contains the data before the perturbation and after the perturbation, are collected to analyse the platoon's stability.

2.4.2 Simulation results

As per the process of (local and string) stability analysis discussed previously, the Taylor expansion coefficients of IDM after linearization are:

$$f_s = \left.\frac{\partial a}{\partial s}\right|_e = \frac{2\alpha}{s_e}\left(\frac{s_0 + Tv_e}{s_e}\right)^2;\ f_v = \left.\frac{\partial a}{\partial v}\right|_e = -\alpha\left[\frac{4}{v_0}\left(\frac{v_e}{v_0}\right)^3 + \frac{2T(s_0+Tv_e)}{(s_e)^2}\right];\ f_{\Delta v} = \left.\frac{\partial a}{\partial \Delta v}\right|_e = \sqrt{\frac{\alpha}{\beta}}\frac{v_e}{s_e}\frac{s_0+Tv_e}{s_e} \tag{53}$$

We can see that the rational driving constraints in Eq. (14) and local stability are easily satisfied since all variables and parameters in Eq. (53) are positive. This is also confirmed by the simulation result because all the simulated vehicles always return to the equilibrium eventually.

The string stability criterion of IDM is shown in Eq. (32).

To assess the influence of each factor in Fig. 2, the string stability region defined by desired time gap T and maximum acceleration α in IDM is plotted for the theoretical result and the simulation result. For each scenario (In total, 89 scenarios were considered), 1600 combinations of desired time gap (0.1-4s) and maximum acceleration (0.1-4m/s^2) are simulated. One example is shown in Fig. 3, which indicates that increasing maximum acceleration or desired time gap enhances the stability of IDM.

To investigate the impact of number of vehicles (i.e., the platoon size) on the overall consistency, we simulate four platoon sizes (i.e., 10, 20, 50 and 100 vehicles) under different disturbances, respectively. Since the equilibrium speed of the platoon is 15 m/s, which implies that a vehicle's largest possible speed decrease is 15m/s, a disturbance with a deceleration of 0.1m/s^2 and a duration of 1s is regarded as a small disturbance; one with a deceleration of 1m/s^2 and a duration of 3s as medium; and one with a deceleration of 2m/s^2 and a duration of 5s as large. The result is shown in Fig. 4.

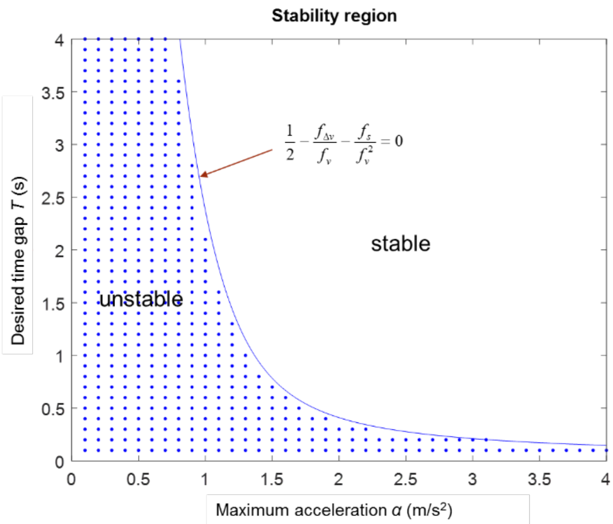


Fig. 3. String stability region for IDM; the solid curve denotes the boundary between the theoretical stable area and the unstable area; the points denote the string unstable parameter sets from simulation; the magnitude of the disturbance in this scenario is 3m/s: the first vehicle decelerated with 1m/s^2 for 3s and then accelerated with 1m/s^2 for 3s.

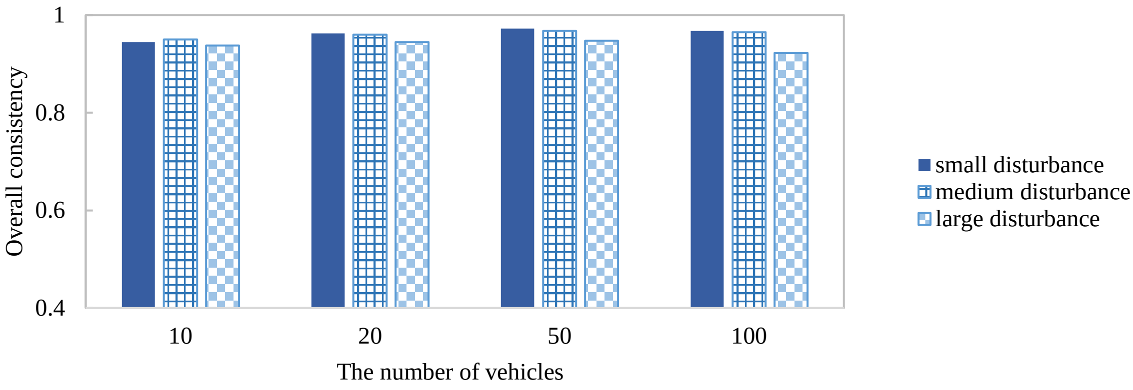
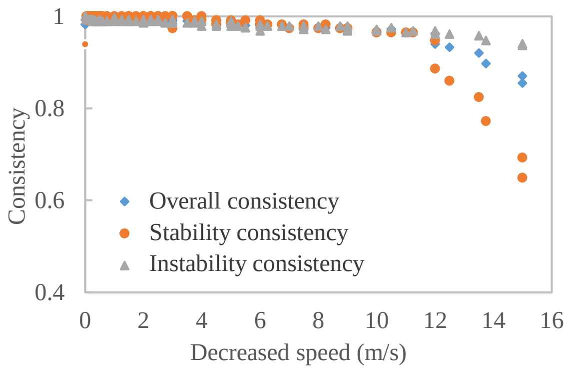


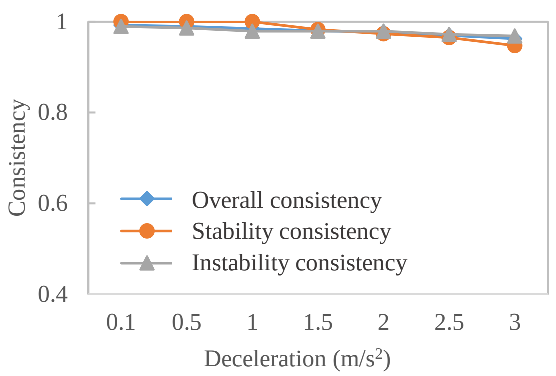
Fig. 4. Platoon size's influence on overall consistency of the string stability results

As shown in Fig. 4, the theoretical result and the simulation result are consistent with each other nicely (above 90%) for different platoon sizes with different disturbances. However, for platoons with a small number of vehicles (e.g., 10), the overall consistency is slightly deteriorated, especially for those scenarios with the small disturbance. This is expected because it is more difficult to reveal from a small number of vehicles the trend of disturbance propagating through the platoon. Thus, to obtain a more reliable result, the platoon size is set as 100 vehicles in the following simulations.

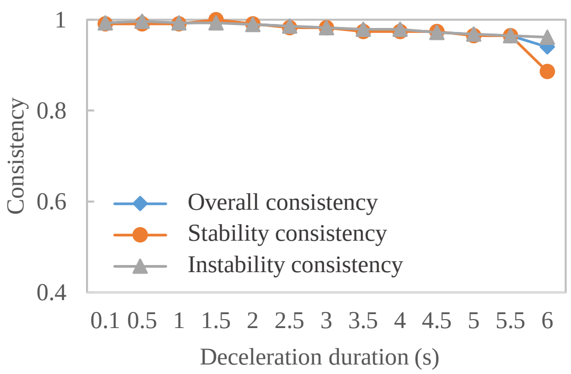
Influence of characteristics (intensity, duration, and magnitude) of the disturbance on the consistency is also evaluated. As indicated in Fig. 2, the range for the intensity is 0.1-3m/s^2, for duration is 0.1-6s and for magnitude is 0.01-15m/s. The simulation result is shown in Fig. 5.



(a) The influence of the disturbance's magnitude (measured by decreased speed)



(b) The influence of the disturbance's intensity (measured by deceleration; and the duration is fixed at 4s)



(c) The influence of the disturbance's duration (measured by deceleration duration; the intensity is fixed at 2m/s^2)

Fig. 5. The disturbance's influence on the stability of IDM

Fig. 5(a) illustrates that the disturbance magnitude has significant impact on the consistency between the theoretical stability analysis and the simulation-based stability analysis. More specifically, when the disturbance magnitude is not large (i.e, the amount of the speed decrease is less than 12m/s), the consistency is high. However, once the disturbance magnitude exceeds 12m/s, the consistency drops rapidly, which means that for a disturbance with a magnitude above 12m/s, linear stability analysis is not applicable anymore. Since the decreased speed is determined by deceleration and duration, we also investigate the effect of deceleration and duration separately by fixing the other one at a constant value. As shown in Fig. 5(b), when the duration is fixed at 4s, the stability consistency is high and does not vary significantly when the intensity (i.e., deceleration) is not large. However, the stability consistency drops notably when the intensity is 2.5m/s^2 or above. As shown in Fig. 5(c), when the intensity is fixed at 2m/s^2, the stability consistency is high and does not vary significantly when the duration is not longer than 5.5s; for a duration longer than 5.5s, the stability consistency drops significantly.

Additionally, we simulated different combinations of the disturbance intensity and duration for the same disturbance magnitude, and did not observe any obvious trend.

3. Stability analysis of TD-CF models

As one of the most important human factors related to driving behaviour, reaction time has been incorporated in CF models since the early stage of traffic flow modelling (Chandler et al., 1958; Saifuzzaman and Zheng, 2014). For traditional vehicles that are completely operated by a human driver, driver's reaction time is the primary source of time delay in CF. However, for connected and/or automated vehicles, time delay can also come from other sources, e.g. delays in sensing, communication, computation, actuation, and etc. (Sharma et al., 2017a). CF models incorporating time delay are named as TD-CF models in this paper. The stability analysis of TD-CF models is generally quite challenging because the time delay factor turns ordinary differential equations into delay-differential equations, which are infinite dimensional systems and difficult to be solved analytically (Zhang and Jarrett, 1997). Several methods mentioned above have been applied for TD-CF models in previous studies, while some new methods also exist, e.g., the Nyquist criterion (Kamath et al., 2015). Notable methods for analysing the local and the string stability of TD-CF models are reviewed in detail in this section.

3.1. TD-CF models

Finite reaction time can be incorporated into CF models by evaluating the acceleration (or speed) at some lagged time $t+\tau$ ($\tau > 0$) with the information available at time *t*. For the sake of simplicity, τ is identical for all variables ($s_n, v_n, \Delta v_n$). While some variants exist in the literature, a basic description of TD-CF models is

$$\dot{v}_n(t) = f(s_n, v_n, \Delta v_n)_{t-\tau} \tag{54}$$

With linearization around the equilibrium point $f(s_e, v_e, 0) = 0$, we obtain

$$\dot{v}_n(t) \approx f(s_e, v_e, 0)_{t-\tau} + f_s(s(t-\tau)-s_e) + f_v(v(t-\tau)-v_e) + f_{\Delta v}(v_{n-1}(t-\tau)-v_n(t-\tau)) \tag{55}$$

Thus the speed variation and the spacing variation are

$$\dot{u}_n(t) \approx (f_s y_n + f_v u_n + f_{\Delta v}(u_{n-1}-u_n))_{t-\tau} \tag{56}$$

$$\dot{y}_n(t) = u_{n-1}(t) - u_n(t) \tag{57}$$

3.2. Local stability analysis

The local stability analysis of TD-CF models is not as popular as the string stability analysis in the literature due to its complexity. Generally, it cannot be solved analytically. Nevertheless, the methods presented below can provide some useful information on local stability properties of TD-CF models.

3.2.1. The Laplace transform based method

Laplace transform can also be used for understanding the local stability properties of TD-CF models, as shown in Eq. (58).

$$se^{s\tau}U_n(s) = \frac{1}{s} f_s\left[U_{n-1}(s) - U_n(s)\right] + f_v U_n(s) + f_{\Delta v}(U_{n-1}(s) - U_n(s)) \tag{58}$$

This equation is similar to Eq. (17) except that an extra term $e^{s\tau}$ appears in the left hand side of Eq. (58) because of the time delay. The transfer function in the frequency domain is thus:

$$U_n(s) = \frac{sf_{\Delta v} + f_s}{e^{s\tau}s^2 - s(f_v - f_{\Delta v}) + f_s} U_{n-1}(s) \tag{59}$$

As introduced before, the characteristic equation for Eq. (59) is $e^{s\tau}s^2 - s(f_v - f_{\Delta v}) + f_s = 0$. Despite its simple appearance, solving this equation for the growth rate $s = \sigma + i\omega$ is nontrivial and can only be done numerically (Treiber and Kesting, 2013).

In Herman et al. (1959), they adopted the Laplace transform based method for a simple linear time-delay model ($\ddot{x}_n(t) = C(\dot{x}_{n-1}(t) - \dot{x}_n(t))_{t-\tau}$ where C is a constant coefficient) and obtained its characteristic equation. The stability criterion $C\tau < \pi/2$ is eventually obtained through some numerical method.

3.2.2. The Nyquist stability criterion

Because of the difficulty of computing the growth rate of perturbations for TD-CF models using either the Laplace transfer based method or the characteristic equation based method, an indirect method, Nyquist stability criterion was proposed in the literature (Kamath et al., 2015) to determine the stability of a closed-loop system by transforming a closed-loop system to an open-loop system. Closed-loop systems and open-loop systems are two common terminologies used in control theory. A control system in which the output and the input are independent is an open-loop system; otherwise, it is a closed-loop system. The main difference between open-loop systems and closed-loop systems is that open-loop systems are open-ended while closed-loop systems has feedback loops between the output and input (thus, closed-loop systems are also called feedback control systems) (Ogata, 2010). Intuitively, CF can be regarded as a closed-loop system. A basic fact of a closed-loop system is that it is stable if all of its poles locate in the left half of the complex plane. This nice feature of a closed-loop system can be used to decide the local stability of TD-CF models, as elaborated below.

Consider a closed-loop system, which has the transfer function:

$$G(s) = P(s)C(s)\big/\big(1 + P(s)C(s)\big) \tag{60}$$

where $P(s)$ and $C(s)$ are rational transfer functions of the loop process and constitutes the open-loop function $L(s) = P(s)C(s)$. The poles of the transfer function are the roots of the denominator, and the stability of the system is determined by the poles, i.e. the roots of $1 + P(s)C(s) = 0$. It is stable when the real part of all the poles are negative. The denominator can be considered as a characteristic equation of the closed-loop system. Breaking the closed-loop system at some point with the open-loop transfer function $L(s) = P(s)C(s)$, while the poles of $L(s)$ are still the poles of $1 + L(s)$, the zeros of $1 + L(s)$ are the poles of $G(s)$. Now the task of stability analysis is to check the number of zeros of the open-loop system in RHP using the Nyquist stability criterion.

Nyquist stability criterion: Given a Nyquist contour Γ_s that encompasses the right half of the complex *s*-plane ($s = \sigma + i\omega$), let *P* be the number of poles of $L(s)$ encircled by Γ_s, and *Z* be the number of zeros of $1 + L(s)$ encircled by Γ_s. Alternatively, and more importantly, *Z* is the number of poles of the closed-loop system $G(s)$ in the RHP. The mapping contour of Γ_s in the $G(s)$ plane, $\Gamma_{L(s)}$ shall encircle (clock-wise) the point $(-1, i0)$ *N* times with $N = Z - P$. (Nyquist, 1932)

From the Nyquist stability criterion, we can see that: if $P = 0$ (no poles of the open-loop function in the RHP) and $N = 0$ (no encirclement of point $(-1, i0)$ by counter $\Gamma_{L(s)}$), $Z = 0$ (no poles in the RHP for the closed-loop function), which means that all the poles of this system locate in the LHP and thus this closed-loop system is stable. Although the stability of a system can be directly determined after a transfer function is found as shown in the section above, the advantage of using the Nyquist stability criterion is that it enables us to quantify the relative stability of a system through the gain margin and the phase margin (since the phase margin is not used in the CF literature, it is not further discussed in this paper), which is particularly important for deriving the stability criterion for TD-CF models, as elaborated below.

First, gain margin g_m is defined as the smallest amount that the open-loop gain (magnitude of the transfer function) can be increased before the closed-loop system becomes unstable. The gain margin can be computed based on the smallest frequency where the phase of the open-loop transfer function $L(s)$ (denoted as $\angle L(s)$) is $-\pi$. This frequency is called the phase crossover frequency, and denoted as ω_{pc}. If the gain magnitude at the phase crossover frequency is less than 1, the encirclement of $(-1, i0)$ is 0. Therefore, to determine the stability of the closed-loop function, we should check the locus of poles of the open-loop function and examine $|L(i\omega_{pc})| < 1$ by finding the phase crossover frequency ω_{pc} (Aström and Murray, 2010).

For TD-CF models, we can consider the CF behaviour between two consecutive vehicles as a closed-loop system with the transfer function shown in Eq. (59). As the characteristic equation is a transformation of open-loop transfer function $L(s)$, $L(s)$ can be also obtained from the characteristic equation:

$$L(s) = \frac{-(f_v - f_{\Delta v})s + f_s}{s^2} e^{-s\tau} \tag{61}$$

The poles of $L(s)$ should not be in the RHP. We then obtain the crossover frequency ω_{pc} at which $\angle L(i\omega_{pc}) = -\pi$ by substituting $s = i\omega$.

$$\angle L(i\omega_{pc}) = \angle\left(-\frac{\left[f_s \cos\omega_{pc}\tau - (f_v - f_{\Delta v})\omega_{pc}\sin\omega_{pc}\tau - i((f_v - f_{\Delta v})\omega_{pc}\cos\omega_{pc}\tau + f_s\sin\omega_{pc}\tau\right]}{\omega_{pc}^2}\right) = -\pi \tag{62}$$

This leads to:

$$\tan\omega_{pc}\tau = -(f_v - f_{\Delta v})\omega_{pc} / f_s \tag{63}$$

At this crossover frequency, $|L(i\omega_{pc})| < 1$ for a stable system, which gives us:

$$((f_v - f_{\Delta v})^2 \omega_{pc}^2 + f_s^2) < \omega_{pc}^4 \tag{64}$$

As Eq. (64) cannot be solved analytically, the local stability criterion for a general TD-CF model cannot be given here. However, using Nyquist stability criterion Kamath et al. (2015) obtained the local stability criterion for the linear CF models proposed in Herman et al. (1959).

3.3. String stability analysis

The reaction time has been considered in CF models since the first string stability analysis study (Chandler et al., 1958), where transfer function is adopted to analyse the string stability of some linear TD-CF models, e.g. response-lag proportionate control, and time-delayed California code (constant time headway) control.

3.3.1. The direct transfer function based method

Similar to the logic shown in Section 2.3.1, the perturbation of the leading vehicle is assumed as $u_0(t) = e^{i\omega t}$. Then $u_n(t) = G^n e^{i\omega t}$. By substituting it into Eq. (56) and (57), we obtain:

$$G(i\omega) = \frac{f_s + i\omega f_{\Delta v}}{-\omega^2 e^{i\omega\tau} - i\omega(f_v - f_{\Delta v}) + f_s} \tag{65}$$

As discussed in Section 2.3.1, the string stability is guaranteed when $|G(i\omega)| < 1$, which can be transformed to:

$$\omega^2\left(\omega^2 + 2\omega(f_v - f_{\Delta v})\sin\omega\tau + f_v^2 - 2f_s\cos\omega\tau - 2f_v f_{\Delta v}\right) > 0 \tag{66}$$

In addition, to ensure that the inequality in Eq. (66) holds for $\omega > 0$, the derivative of the left-hand side (a function of ω) should be positive when $\omega \to 0$, which leads to:

$$2\omega + 2(f_v - f_{\Delta v})\sin\omega\tau + 2\omega\tau(f_v - f_{\Delta v})\cos\omega\tau + 2\tau f_s \sin\omega\tau > 0 \tag{67}$$

As $\omega \to 0$, we have $\sin\omega\tau \approx \omega\tau$, $\cos\omega\tau \to 1$. Then Eq. (66) and (67) can be simplified as:

$$f_v^2 - 2f_s - 2f_v f_{\Delta v} > 0 \tag{68}$$

$$1 + 2\tau(f_v - f_{\Delta v}) + \tau^2 f_s > 0 \tag{69}$$

By inserting Eq. (68) into Eq. (69), the string stability criterion of TD-CF models is:

$$\frac{1}{2}-\frac{f_{\Delta v}}{f_v}-\frac{f_s}{f_v^2}\geq 0 \,\&\, 1+2\tau(f_v-f_{\Delta v})+\tau^2(\frac{1}{2}f_v^2-f_v f_{\Delta v})>0 \tag{70}$$

As shown above, the stability criterion for TD-CF models is more complex than that for B-CF models, which make it more difficult to be satisfied. In other words, reaction time tends to destabilise the CF behaviour. Chandler et al. (1958) and Ferrari (1994) utilised the direct transfer function based method and the Laplace transform based method to investigate the string stability properties of several linear TD-CF models. Bando et al. (1998) also applied this method to analyse stability of the time-delayed optimal velocity model (OVM) (Bando et al., 1995).

Note that the Laplace transform based method can also be used to solve the string stability of TD-CF models (Orosz et al., 2011). The procedure is similar to that in Section 2.3.2. Thus, it is not discussed here to avoid duplication.

3.3.2. The characteristic equation based method

As shown in Section 2.3.3, the characteristic equation based method is a general approach for analysing string stability. This method has also been applied to study the string stability properties of TD-CF models, although the process is generally very complex, as briefly described below.

Inserting the Fourier ansatz of spacing variation and speed variation $u_n=e^{\lambda t+in\varphi}\hat{u}$, $y_n=e^{\lambda t+in\varphi}\hat{y}$ into Eq. (56) and (57) gives us

$$\begin{pmatrix}\lambda & (1-e^{-i\varphi}) \\ -f_s & \lambda e^{\lambda\tau}-(f_v-f_{\Delta v}+f_{\Delta v}e^{-i\varphi})\end{pmatrix}\begin{pmatrix}\hat{y} \\ \hat{u}\end{pmatrix}=0 \tag{71}$$

A TD-CF model is string stable if the determinant of the matrix of coefficients is equal to zero, which leads to a quadratic characteristic equation:

$$\lambda^2-\left[(f_v+f_{\Delta v}-f_{\Delta v}e^{-i\varphi})e^{-\lambda\tau}\right]\lambda+\left[f_s(1-e^{-i\varphi})e^{-\lambda\tau}\right]=0 \tag{72}$$

The characteristic equation can be transformed into a three-order equation by expanding the exponential terms of delay to the second order, i.e., $e^{-\lambda\tau}=1-\lambda\tau+(\lambda\tau)^2/2+\ldots$. Then the solutions of this characteristic equation is extracted to evaluate the string stability. More specifically, if the real parts of the solutions are all negative, it is string stable; otherwise, it is string unstable. Due to its complexity, the mathematical detail of extracting the roots is omitted here. For more information, see Ngoduy (2013).

Based on the characteristic equation, and using the Hopf bifurcation method presented in Section 2.3.3, Orosz et al. (2010a, 2011) separated the real part and imaginary part results in the Hopf bifurcation curves for f_s/f_v^2 and τf_v. Based on the results in Orosz et al. (2010a), we obtain a stability criterion for time-delayed CF models:

$$\frac{1}{2}-\frac{f_{\Delta v}}{f_v}-\frac{f_s}{f_v^2}\geq 0 \,\&\, \frac{\pi}{2}-\tau f_v(\frac{\pi}{2}+1)-2\tau>0 \tag{73}$$

Obviously, the stability criterion depicted in Eq. (73) is different from that shown in Eq. (70). Both the criteria are evaluated in the numerical simulation.

3.4. Numerical simulation

The same simulation setup for B-CF models is used here. To keep the consistency of simulation analysis through the paper, IDM is also adopted here by introducing a constant reaction time to form a time-delayed IDM (TD-IDM), as mathematically defined in Eq. (74).

$$\dot{v}_n(t)=\alpha\left[1-\left(\frac{v_n(t-\tau)}{v_0}\right)^4-\left(\frac{s_n^*(t-\tau)}{s_n(t-\tau)}\right)^2\right] \tag{74}$$

where τ is the reaction time of a human driver. Other variables are the same as in Eq. (49).

The Taylor expansion coefficients of TD-IDM are identical with those of IDM. Two different theoretical stability criteria of TD-IDM can be obtained from using the Laplace transform based method (Eq. (70)) and from using Hopf bifurcation method (Eq. (73)), respectively. Similar to what is presented in Section

2.4, the string stability region is drawn for the simulation result and the theoretical result, as shown in Fig. 6. This figure clearly shows the reaction time's negative impact on the stability of TD-IDM, because the stable region is significantly reduced, compared with that of IDM, which is probably due to the high-frequency disturbances caused by reaction time (Treiber et al., 2007). In addition, the theoretical result from the Laplace transform based method performs much better than that from Hopf bifurcation method. The larger inconsistency of the criterion from the Hopf bifurcation method is likely to be caused by the approximation of results in Orosz et al (2010a) since it is not possible to obtain the results directly.

Similarly, for TD-IDM, we first study the influence of disturbance on the consistency between the simulation result and two theoretical results with a fixed reaction time 0.5s. As shown in Fig. 7, and trends similar to those for IDM are observed. The consistency between the simulation result and the theoretical results from both the methods start to decline when the decreased speed exceeds 8m/s, which is lower than that for IDM (12m/s). Again, this implies the negative impact of reaction time on the applicability of theoretical stability analysis. In addition, the comparison of Fig. 7(a) and Fig. 7(b) further confirms the better performance of the theoretical result from the Laplace transform based method. Thus, for the impact of disturbance intensity and disturbance duration, we only present the analysis result using the theoretical stability criterion from the Laplace transform based method. As shown in Fig. 7(c), when the duration is fixed at 4s, the stability consistency does not vary significantly when the intensity is not large. However, the stability consistency drops notably when the intensity is $2m/s^2$ or above. As shown in Fig. 7(d), when the intensity is fixed at $2m/s^2$, the stability consistency does not vary significantly when the duration is not longer than 4s; for a duration longer than 4s, the stability consistency drops significantly. Compared with the results from the simulation for IDM, it is clear that disturbance's impact on the consistency of the theoretical results for TD-IDM is more significant across all the aspects (disturbance intensity, duration, and magnitude). In addition, even for small disturbances, the consistency of the theoretical results for TD-IDM's stability is notably worse than that for IDM.

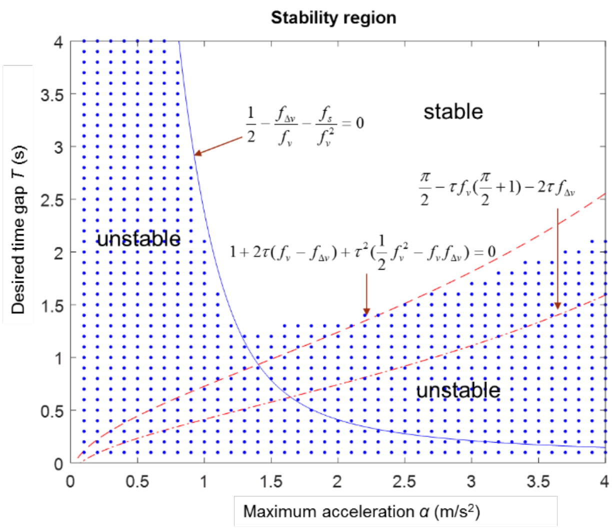
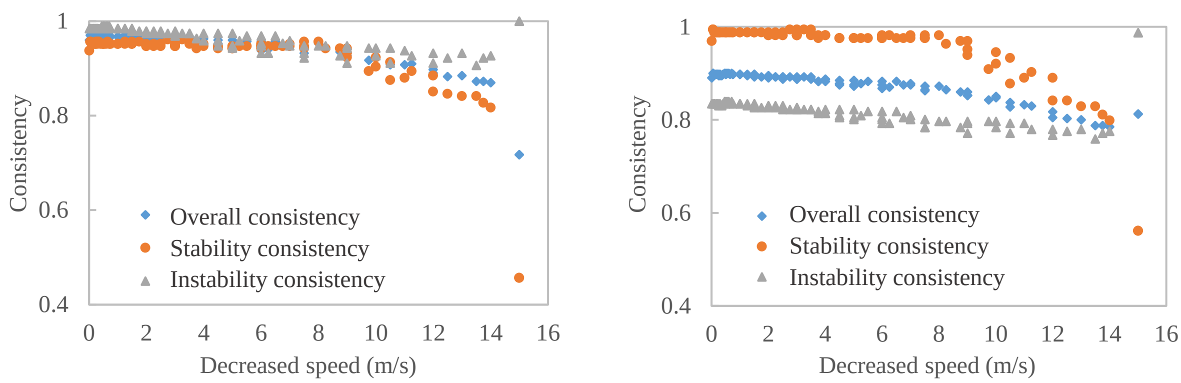
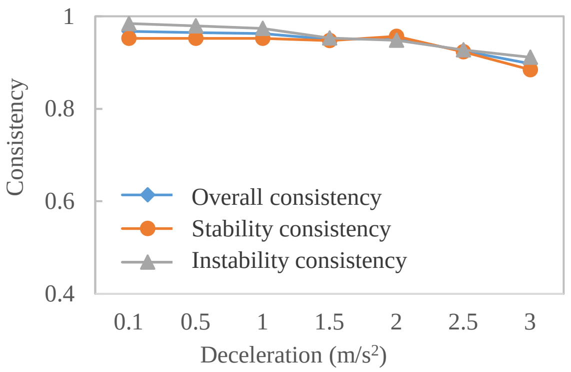


Fig. 6. String stability region for TD-IDM; the solid line and dashed line denote the boundary of the theoretical criterion from the Laplace transform based method; the solid line and dash-dot line denote the boundary of the theoretical criterion from the Hopf bifurcation method; the points denote the string unstable parameter sets from simulation for TD-IDM with reaction time of 0.5s; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with $1m/s^2$ for 3s and then accelerates with $1m/s^2$ for 3s.

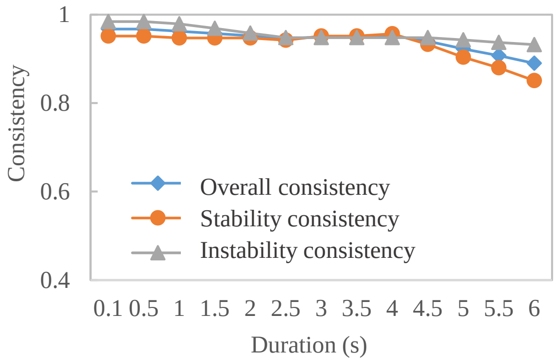


(a) The influence of the disturbance's magnitude compared with the theoretical result of using the Laplace transform based method (measured by decreased speed)

(b) The influence of the disturbance's magnitude compared with the theoretical results of using the Hopf bifurcation method (measured by decreased speed)



(c) The influence of the disturbance's intensity (measured by deceleration, and the decelerating duration is fixed at 4s)



(d) The influence of the disturbance's duration (measured by deceleration duration, and the deceleration is fixed at $2m/s^2$)

Fig. 7. The influence of disturbance on the string stability of TD-IDM

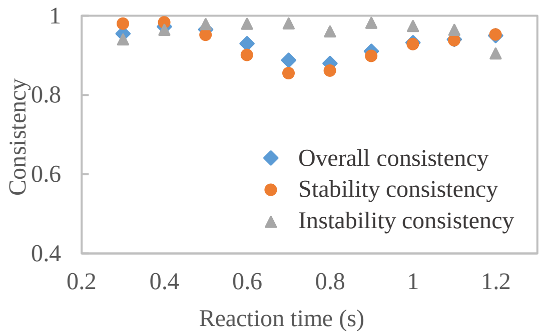


Fig. 8. The influence of reaction time on the string stability of TD-IDM; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with $1m/s^2$ for 3s and then accelerates with $1m/s^2$ for 3s.

Since the reaction time is the critical feature of TD-IDM, we also investigated the impact of different reaction times on the consistency of simulated and theoretical string stability results with a fixed disturbance. As discussed above, only the comparison result for using the Laplace transform based method is presented in Fig. 8. As shown in this figure, the overall consistency and the stability consistency first decrease and then increase with the increase of reaction time, while the instability consistency shows no obvious trend.

For a general TD-CF model, although no explicit theoretical criterion exists in the literature for local stability, the stability region can be obtained using the simulations. Different from the simulation of string stability, for local stability we only check the speed variation of the second vehicle. If the speed of the second vehicle eventually converges to the equilibrium speed, it is locally stable. An example of the local stability region is plotted in Fig. 9 by simulating different combinations of the desired time gap and the maximum acceleration. Fig. 9 illustrates that, unlike IDM whose local stability is guaranteed, there exist locally unstable parameter sets for TD-IDM. For the convenience of comparison, the string stability region obtained by using the Laplace transform based method is also plotted in Fig. 9. It clearly shows that all the locally unstable parameter sets locate in the string unstable area, which is consistent with the definitions of local stability and string stability. Although in the human driving behaviour, local stability is always satisfied, our simulation result implies that inadvertent parameter selections may cause local instability of TD-CF models. Particularly, when the time delay is 0.5s, a small desired time gap around 0.5s that is often assumed for automated vehicles (Ge and Orosz, 2014; Ngoduy, 2015; Sharma et al., 2017a), is more likely to make the TD-IDM locally unstable.

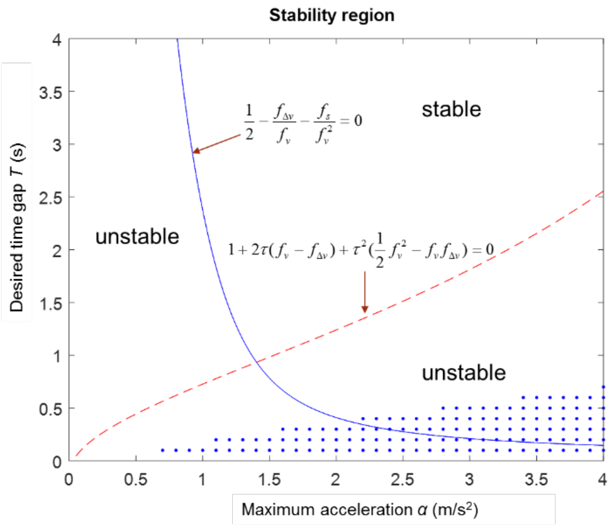


Fig. 9. Local stability region of TD-IDM; the solid line and dashed line denote the string stability boundaries; the points denote the locally unstable parameter sets for TD-IDM with a reaction time of 0.5s; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with 1m/s^2 for 3s and then accelerates with 1m/s^2 for 3s.

4. Stability analysis of MAC-CF models

Anticipation is another human factor that has been frequently considered in the literature by postulating that human drivers achieve additional stability and safety by taking into account next-nearest neighbours and/or further vehicles ahead (Treiber et al., 2006). CF models considering vehicles more than the immediate leading one were studied in Herman et al. (1959) for the first time. Multi-anticipation becomes more relevant in a connected environment where a vehicle is capable of cooperating with other vehicles within the communication range. In general, considering more than two vehicles in one acceleration function complicates the computation of characteristic equation or transfer function for MAC-CF models. Besides, in MAC-CF models, delays caused by a human driver's reaction time and the connected environment's communication time are sometimes considered simultaneously. Therefore, the difficulty of stability analysis for MAC-CF is even greater than that for TD-CF models discussed above.

4.1. MAC-CF models

Multi-anticipative CF models describe the motion of vehicle *n* in relation to its leading vehicles (*n*–*m*) where *m*=1, 2, 3, …, *M*. Similarly, the cooperative CF models incorporate the information of the leading vehicle *n*-1 and *m* connected vehicles ahead where $m \in [2, M]$. Two forms of MAC-CF models were generalised in the literature (Sau et al., 2014): one adopts relative positions and speeds between successive vehicles (the MAC-IDM proposed in Section 4.3 also uses this form); the other employs relative positions and speeds between the subject vehicle and its leaders (the leaders can be distributed discretely in a platoon), and its general form of the acceleration of vehicle *n* as:

$$\dot{v}_n(t) = f\left(v_n(t-\tau^v), \sum_m a_m s_{n,n-m}(t-\tau^s_{n,n-m}), \sum_m b_m \Delta v_{n,n-m}(t-\tau^{\Delta v}_{n,n-m})\right) \tag{75}$$

where $s_{n,n-m}$ and $\Delta v_{n,n-m}$ denotes the gap and relative speed between vehicle *n* and its leader *n-m*, respectively. In this formulation, time delays in sensing relative gap (i.e. $\tau^s_{n,n-m}$) and relative speeds (i.e. $\tau^{\Delta v}_{n,n-m}$) with respect to its leader *n-m* vary, and a_m and b_m represent weighting factors for the relative gap and relative speed, which generally satisfies $a_1 > a_2 > ... > a_M$, $b_1 > b_2 > ... > b_M$. The delay caused by sensing/computing *n*th vehicle's own speed τ^v is also considered in the general model.

The linearization of Eq. (75) around the equilibrium point $f(s_e, v_e, 0) = 0$ leads to:

$$\dot{y}_{n,n-m}(t) = u_{n-m}(t) - u_n(t) \tag{76}$$

$$\dot{u}_n(t) \approx f_s \sum_m a_m y_{n,n-m}(t-\tau^s_{n,n-m}) + f_v u_n(t-\tau^v) + f_{\Delta v} \sum_m b_m \left(u_{n-m}(t-\tau^{\Delta v}_{n,n-m}) - u_n(t-\tau^{\Delta v}_{n,n-m})\right) \tag{77}$$

Since no previous studies focused on the local stability of MAC-CF, no theoretical analysis on the local stability is presented here (but it has been investigated by using the simulation as discussed later). Instead, we focus on the previous attempts of string stability analysis for MAC-CF.

4.2. String stability analysis

4.2.1. The characteristic equation based method

As the characteristic equation based method is a general approach for string stability analysis, it is also frequently adopted for investigating string stability properties of MAC-CF. Inserting the Fourier-ansatz of spacing variation and speed variation $u_n = e^{\lambda t + in\varphi}\hat{u}$, $y_n = e^{\lambda t + in\varphi}\hat{y}$ into Eq. (76) and (77) gives us:

$$\lambda^2 + \left[f_v e^{-\lambda\tau^v} - f_{\Delta v}\sum_m b_m (e^{-i\varphi m} - 1) e^{-\lambda \tau^{\Delta v}_{n,n-m}} \right]\lambda + \left[f_s (e^{-i\varphi m} - 1)\sum_m a_m (e^{-i\varphi m} - 1) e^{-\lambda\tau^{s}_{n,n-m}} \right] = 0 \tag{78}$$

The system is string stable if all the real parts of the solutions of Eq. (78) are negative. The solution λ can be expanded to a power series $\lambda = i\varphi\lambda_1 + \varphi^2\lambda_2 + \ldots$ where λ_1 and λ_2 are real coefficients. Substituting the expanded λ into Eq. (78), corresponding stability criterion can be solved after a rather lengthy process. See Ngoduy (2015) for the detail of a generized multi-anticipative CF model, which obtains the criterion:

$$\left(\frac{f_v}{f_s}\right)^2 \left(\sum_m a_m - \frac{1}{2}\right) + \frac{f_v f_{\Delta v}}{f_s^2}\sum_m b_m m - \frac{1}{f_s} + \frac{f_v}{f_s}\sum_m a_m \tau^s_{n,n-m} > 0 \tag{79}$$

Monteil et al. (2014) also obtained the characteristic equation of a cooperative CF model without solving it.

As discussed in Section 2.3.4, the roots of the quadratic characteristic equation can also be plotted on a complex plane via the root locus method. This strategy was employed in Sau et al. (2014) to analyse some cooperative strategies.

Meanwhile, the characteristic equation based method was combined with the Routh-Hurwitz criterion (see Section 2.2.1) in Lenz et al. (1999) to investigate the string stability of MAC-OVM.

4.2.2. The Laplace transform based method

Due to the uncertainty of cooperative vehicles in a platoon (e.g., not all the vehicles in the communication range are connection-enabled), analytically solving the characteristic equation for a cooperative CF model can be difficult. An alternative is to obtain the head-to-tail transfer function of a platoon via Laplace transform, as explained below. A platoon is considered as being stable when the perturbations are attenuated when they propagate from the head to the tail (the evolution of perturbations within the platoon is ignored) (Ge and Orosz, 2014).

Consider a platoon that contains *n* vehicles, among which *m+1* vehicles are connected (i.e., the *n*th vehicle plus *m* vehicles ahead). By transforming MAC-CF (Eq. (76) and (77) for connected vehicles) and TD-CF model (Eq. (56) and (57) for manual vehicles) from the time domain to the frequency domain, we get:

$$sU_n(s) = \frac{1}{s} f_s \sum_m \alpha_m \left[U_{n-m}(s) - U_n(s)\right] e^{-s\tau^s_{n-m+1}} + f_v U_n(s) e^{-s\tau^v} + f_{\Delta v}\sum_m \beta_m (U_{n-m}(s) - U_n(s)) e^{-s\tau^{\Delta v}_{n-m+1}} \tag{80}$$

$$se^{s\tau}U_n(s) = \frac{1}{s} f_s \left[U_{n-1}(s) - U_n(s)\right] + f_v U_n(s) + f_{\Delta v}(U_{n-1}(s) - U_n(s)) \tag{81}$$

The head-to-tail transfer function of perturbations in the frequency domain is then obtained as $G(s) = U_1(s)/U_n(s)$.

Therefore, the necessary and sufficient condition for the head-to-tail string stability is given by $\left|G(i\omega)\right| < 1$. Apparently, the general head-to-tail transfer function is rather difficult to obtain and can only be analysed for specific conditions. Moreover, the magnitude of such transfer function cannot be solved analytically, and its numerical solutions with different critical frequencies can be obtained via some computational tools, e.g., the continuation package DDE-BIFTOOL (Engelborghs et al., 2001).

The Laplace transform based method was adopted by Liu et al. (2001); Swaroop (1997); Swaroop and Hedrick (1996) for analysing the string stability of certain cooperative CF models.

4.3. Numerical simulation

Because of the huge importance of CAV, to gain more insights on CAV's impact on traffic flow, two types of MAC-IDM are simulated. First, for consistency, the MAC-IDM without reaction time which is proposed by Treiber et al. (2006) is used for simulation, and its mathematical equations are shown below:

$$\dot{v}(t) = \alpha \left[1 - \left(\frac{v_n(t)}{v_0} \right)^4 - \left(\frac{s^*(t)}{\sum_{m=1}^{M} \gamma_m s_{n-m+1}(t)} \right)^2 \right] \tag{82}$$

$$s^*(t) = s_0 + Tv_n(t) - \frac{v_n(t) \sum_{m=1}^{M} \gamma_m \Delta v_{n-m+1}(t)}{2\sqrt{\alpha\beta}} \tag{83}$$

where $\Delta v_{n-m+1}(t) = v_{n-m}(t) - v_{n-m+1}(t)$, $s_{n-m+1}(t) = x_{n-m}(t) - x_{n-m+1}(t) - l_{veh}$ are the relative speed and bumper-to-bumper gap of two consecutive vehicles, respectively; and γ_m is the weight of different leading vehicles with $\gamma_1 > \gamma_2 > ... > \gamma_M, \sum_{m=1}^{M} \gamma_m = 1$.

As there is no reaction time in this type of MAC-IDM, the local stability of this model is expected to be always satisfied, which is confirmed by the simulation (no detail is presented here to save space).

The string stability criterion of MAC-IDM can be derived from Sau et al. (2014) as below.

$$\frac{1}{2} - \frac{f_{\Delta v}}{f_v} - \frac{f_s}{f_v^2} + \sum_{m=1}^{M} (m-1)\gamma_m \geq 0 \tag{84}$$

For simplicity, we just consider two vehicles ahead. With the same simulation setup as for IDM and TD-IDM, the string stability regions from the simulation result and from the theoretical result are shown in Fig. 10. For the convenience of comparison, the boundary for IDM is also added (i.e., the solid curve in the figure). This figure clearly shows that multi-anticipation enhances the string stability of traffic flow.

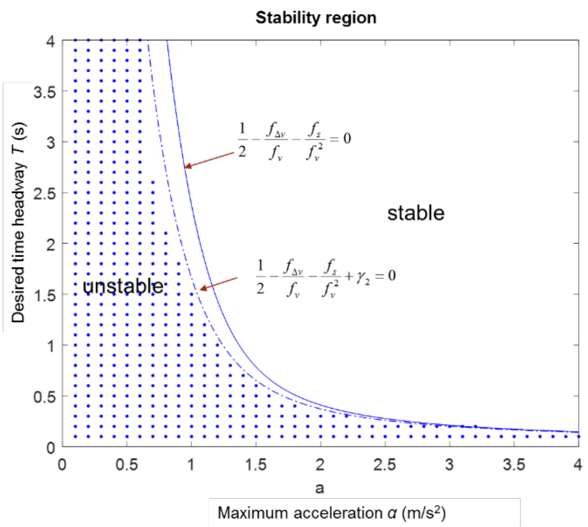


Fig. 10. String stability region for MAC-IDM; the solid curve denotes the boundary of the theoretical criterion for IDM; the dash-dot curve denotes the boundary of the theoretical criterion for MAC-IDM; the points denote the string unstable parameter sets from simulation for MAC-IDM with $\gamma_1 = 0.7, \gamma_2 = 0.3$; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with 1m/s^2 for 3s and then accelerates with 1m/s^2 for 3s.

For disturbance's influence on the consistency of the theoretical result for MAC-IDM's stability, observations similar to those for IDM and TD-IDM can be obtained. To save space, they are not repeated here. Overall, impact of the disturbance intensity, duration, and magnitude on MAC-IDM's stability consistency is closer to that observed for IDM.

Compared with IDM and TD-IDM, a unique feature of MAC-IDM is weights for all the leading vehicles. Thus, weights' impact on the consistency of the theoretical result for MAC-IDM's stability is investigated. As shown in Fig. 11, the overall consistency and stability consistency first decrease and then increase as the weight for the first leading vehicle increases. Such finding can have important implication on developing control strategies for connected vehicles because inappropriately assigned weights to different vehicles can cause the deterioration of the stability, and thus the loss of the effectiveness of connectivity, because a larger inconsistency with one weight means a higher chance of failing to achieve the expected control effects using the theoretical result.

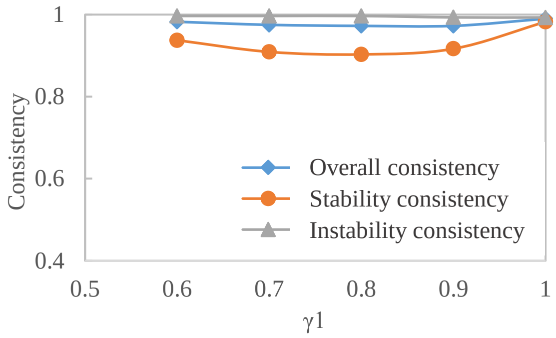


Fig. 11. Weight's influence on the string stability of MAC-IDM; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with 1m/s^2 for 3s and then accelerates with 1m/s^2 for 3s; when γ_1=1, MAC-IDM becomes B-IDM.

Meanwhile, to be more realistic and complete, another type of MAC-IDM (CAV-IDM) is simulated by incorporating time delay into Eq. (82):

$$\dot{v}(t+\tau)=\alpha\left[1-\left(\frac{v_n(t)}{v_0}\right)^4-\left(\frac{s^*(t)}{\sum_{j=1}^{m}\gamma_j s_{n-j+1}(t)}\right)^2\right] \tag{85}$$

where τ is the time delay of CAVs caused by sensing, processing, and etc.

Due to the absence of a proper stability criterion for CAV-IDM in the literature, we analyse its stability numerically with the same simulation setup for MAC-IDM. Compared with TD-IDM, the local stability of CAV-IDM is significantly enhanced thanks to the introduction of connectivity (the figure is not presented here to save space because only a few unstable points are obtained in the simulation). This model's string stability region from the simulation result is shown in Fig. 12. For the purpose of comparison, the string stability criterion for TD-IDM with the same amount of time delay obtained by using the Laplace transform based method is also plotted in Fig. 12.

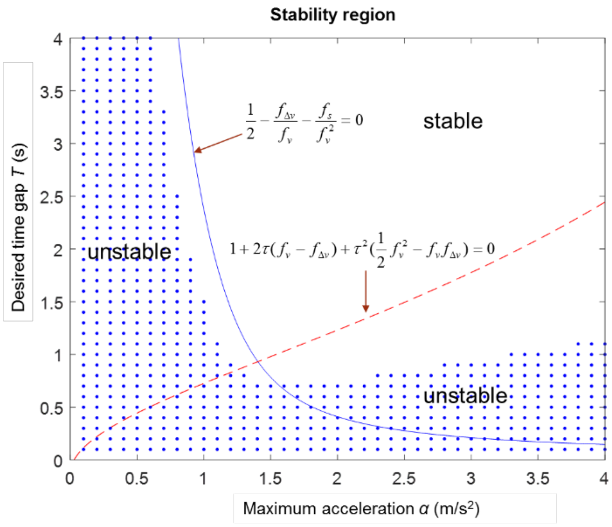


Fig. 12. String stability region for CAV-IDM; the solid and dashed lines denote the boundary of the theoretical criterion for TD-IDM with a time delay of 0.5s from the Laplace transform based method; the points denote the string unstable parameter sets from simulation with $\gamma_1=0.7,\gamma_2=0.3$ and reaction time of 0.5s; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with 1m/s^2 for 3s and then accelerates with 1m/s^2 for 3s.

Similar to the results for MAC-IDM, Fig. 12 reveals that the connectivity significantly improves the string stability of CAV-IDM compared with TD-IDM. To further study the impact of time delay and connectivity on the string stability of CF models, we conduct a sensitivity analysis on the instability with different time delays and weights for CAV-IDM, while the disturbance is fixed. The result is shown in Fig. 13.

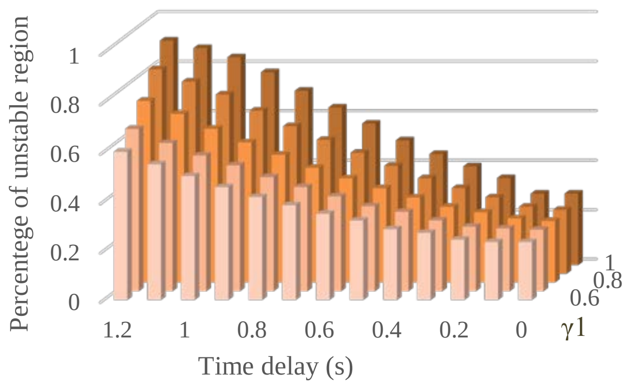


Fig. 13. Sensitivity analysis result for CAV-IDM; the disturbance magnitude in this scenario is 3m/s: the first vehicle decelerates with $1m/s^2$ for 3s and then accelerates with $1m/s^2$ for 3s; when time delay is 0 and γ1 is 1, it becomes B-IDM.

As shown in Fig. 13, the time delay and connectivity both significantly affect the string stability of CAV-IDM , while they play totally opposite roles: increasing time delay quickly reduces the stability, while a stronger connectivity (as indicated by the growing weight of the second leading vehicle) improves the stability. Additionally, by comparing the percentages of unstable region before and after incorporating connectivity, connectivity's positive impact on stability first rapidly increases and then gradually decreases with the increase of time delay; and it reaches the peak at a time delay of 1s (the figure is not included here to save space). Overall, despite the possible undesirable effect of suboptimal weights assigned to different vehicles, connectivity seems critical to the maintenance of CAVs' stability in the presence of time delay.

5. Discussion

5.1. Summary of linear stability analysis methods

This section discusses important features (including strengths and issues) of each method reviewed in the paper for analysing linear stability of different types of CF models. Note that the name of each stability method reviewed in the paper is given by the authors, primarily for the convenience of discussion, and overlapping still exists in this taxonomy. Particularly, although one method is called the direct transfer function based method in this paper because of the central role of transfer function, transfer function also appears in some other methods, e.g., the Laplace transform method and the Nyquist stability criterion. Also the direct transfer function based method is the base of the ansatz used in the characteristic equation method for string stability analysis in Section 2.3.3.

As a general approach, the characteristic equation based method is based on the characteristic equation induced from linearized CF models with the use of exponential ansatz to represent perturbations (the rind road configuration analysis can also be used to derive the characteristic equation in string stability analysis). After obtaining the characteristic equation whose roots contain information on the growth rate of perturbations, several techniques are capable of finding the stability criterion: The root extracting technique through Taylor expansion; the root locus method; the Routh-Hurwitz criterion; Nyquist criterion; and the Hopf bifurcation method. These techniques exhibit different features when analysing the stability of CF models. The growth rate of perturbation is obtained directly via root extracting of the characteristic equation. However, it is very complicated (sometimes insolvable) to solve the equation for some complicated CF models (e.g. time-delayed CF models, MAC-CF models). In contrast, the root locus method is quite efficient for such complex CF models and can show the stability in a graphical way. However, it cannot give the stability criterion by itself. Moreover, it can only tell the trend of stability with varying specific parameters. The Routh-Hurwitz criterion is a direct and simple technique for simple CF models with no need to solve the characteristic equation. However, it is not suitable for a complicated characteristic equation with complex coefficients. The Nyquist criterion is a good alternative for complex CF models (especially time-delayed CF models), although the computation is rather complex sometimes. The Nyquist criterion is generally not suitable for string stability analysis whereas the other techniques are. The Hopf bifurcation method was only implemented for string stability analysis in the literature. However, there should be no problem to extend it for local stability analysis, although as shown in our simulation, the string stability criterion of TD-IDM derived with this method has a relatively poor consistency.

Lyapunov stability criterion is a method frequently used for stability analysis in control theory, but has not attracted much attention in the CF literature. Using Lyapunov stability criterion, there is no need to derive or solve the characteristic equation, which significantly simplifies the stability analysis. Additionally, Lyapunov stability criterion can be a powerful tool for overcoming the challenges related to the complexity of stability analysis of TD-CF models because of its good extendibility, e.g. Lyapunov-Razumikhin approach (Gu and Niculescu, 2003). However, to use Lyapunov stability criterion, considerable skills are needed in constructing a Lyapunov function. Although there are several general methods (e.g. Krasovki theorem), constructing a suitable Lyapunov function can still be challenging. Moreover, while the existence of a Lyapunov function can be used as a proof of the system's stability, the non-existence of a Lyapunov function does not necessarily imply the system's instability. That is, the Lyapunov stability criterion is only a sufficient condition of a system's stability, but not a necessary condition. However, this should not be a big concern in practice because by using the specific quantitative stability criterion derived from this method, control strategies that can guarantee the system's stability can be developed.

The direct transfer function based method and the Laplace transform based method are two similar but still different approaches for stability analysis. The main similarity is that the stability criterion from both the methods is obtained according to the magnitude of transfer function. However, one main difference is that the transfer function of direct transfer function based method is obtained directly by assuming the perturbations as exponential terms in the time domain whereas for the Laplace transform based method the transfer function is obtained in the frequency domain, which is a much easier way. Another main difference is that the Laplace transform based method can also be used for local stability analysis in combination with the characteristic equation based method by transforming the transfer function from the frequency domain to the time domain, while the direct transfer function based method is just for string stability analysis. Compared with other approaches, these two approaches are computationally effective for time-delayed CF models. However, for MAC-CF models, they appear to be powerless.

A summary of the stability analysis methods reviewed in this paper is presented in Table A2.

5.2. Issues of linear stability analysis

A summary of representative studies of stability analysis of CF models is presented in Table A1. Although researchers started implementing the stability analysis of various CF models since the early days of traffic modelling, CF model stability analysis' practical implications are largely ignored. In other words, stability analysis of CF models have not been fully utilised to guide the development of traffic operations and control strategies. Although stability analysis is routinely implemented for developing better control strategies in the control theory to maximize the system's stability, it is still yet to become a norm in traffic modelling, where stability analysis is often just an intellectual exercise of traffic flow modelling theorists while it is almost totally ignored by practitioners. There is much we can learn from how stability analysis is used in other disciplines, control theory in particular. This is partially caused by the technical barriers because all the stability analysis techniques are developed in other disciplines with high mathematical complexity. This is one of the main motivations of this paper.

Meanwhile, to use stability analysis to guide the development of traffic control strategies, the consistency, applicability and robustness of the stability analysis result needs to be sufficiently high. Checking important factors' impact on the consistency between the theoretical result of stability analysis and the simulation result can shed light on the value of stability analysis in practice, e.g., if the consistency is sensitive towards the platoon size, that would imply that stability analysis has limited value in practice because the platoon size in real traffic naturally varies significantly. This is another motivation of this paper. From our numerical simulation analysis, we can see that the inconsistency between the theoretical result and simulation result indeed exists. Such inconsistency are influenced by several factors, among which reaction time is perhaps the most significant one that affects the stability analysis—not only the complexity of theoretical analysis, but also the inconsistency between the theoretical result and the simulation result.

An unrealistic assumption that underlines the stability analysis is that the platoon is assumed to be in an equilibrium state throughout the entire analysis except some small disturbance added by the analyst. Although such assumption's merit is obvious from analytical perspective, it can make stability analysis results less applicable to the real traffic. In a real CF system, a vehicle can be in different driving regimes at different times, such as free flow, accelerating, decelerating, cruising, and stand still (see Sharma et al. (2017b) for the definitions of each driving regime). Even if the stability of CF model is promised with regard to small perturbations in an equilibrium state, its validity is questionable if vehicles in a platoon are experiencing other driving regimes. Therefore, to increase stability analysis' practical value, different driving regimes need to be considered to better mimic real-world traffic phenomena. Such work is ongoing.

Another important limitation of linear stability analysis of CF models is rooted in the fact that most of CF models are not linear so that they have to be linearized before a linear stability analysis can be implemented, which inevitably causes approximation errors. That is, the result accurate for the linear approximation of CF models may not be accurate for the original system.

Similarly, an inherent limitation of linear stability analysis is that linear stability analysis is only suitable for capturing impact of small perturbations. For large perturbations that are ubiquitous in real traffic (e.g. inconsiderate hard braking or lane changes, see Zheng 2014 for a review), their nonlinear effects on surrounding traffic generally should not be ignored. Because of its complexity, only a few studies in the literature implemented nonlinear stability analysis on traffic flow, e.g., Monteil et al. (2014) studied the nonlinear shock wave for general CF models by deriving the Korteweg-de-Vries (KdV) equations through the reductive perturbation method. Orosz et al. (2010b) investigated the nonlinear dynamics of stop-and-go wave using OVM. In addition, to simplify the problem, some of these studies focused on macroscopic models instead of CF models, e.g., Ge et al. (2005) adopted the modified Korteweg-de-Vries (mKdV) equation to describe the nonlinear traffic wave near the critical point, using the reductive perturbation method on macroscopic models. However, as pointed out by Treiber and Kesting (2013), the conditions for deriving equations for shock waves (e.g., KdV or mKdV equation) are extremely restrictive and nearly never satisfied in real traffic situations. Alternatively, numerical simulation with a closed single-lane ring road is a simple way to investigate the nonlinear stability of traffic flow, from which a stability diagram that shows different stability regions in the full range of average density can be easily obtained (Treiber and Kesting, 2013).

Recently, CAVs have attracted lots of attention both from researchers and practitioners because CAVs is expected to revolutionize how people travel and how our transportation systems operate, which can ultimately solve many issues caused by the current transportation systems, e.g. road safety, traffic congestion, vehicle fuel consumption and emissions, and etc. Particularly, our analysis shows that time delay and connectivity can significantly impact traffic flow's stability in opposite directions, and connectivity is critical for maintaining CAVs' stability in the presence of time delay. In our view, utilising stability analysis and developing effective control strategies accordingly by taking advantage of the connected environment is a promising direction and worth further study, e.g., according to the simulation result of MAC-IDM, we can get a better idea on how to properly assign weights to different leading vehicles within the communication range when designing control strategies.

6. Conclusions

This paper comprehensively reviews and assesses major methods for analysing local and string stability for three types of car-following (CF) models: basic, time-delayed, and multi-anticipative/ cooperative CF models. For each type of CF models, notable methods in the literature for analysing its local stability and string stability have been reviewed in detail, including the characteristic equation based method (e.g., root extracting, root locus method, the Routh-Hurwitz criterion, the Nyquist criterion and the Hopf bifurcation method), Lyapunov criterion, the direct transfer function based method, and the Laplace transform based method. To demonstrate how each method can be exactly applied to CF models, IDM and its time-delayed and multi-anticipative/cooperative variants are used.

Moreover, consistency and applicability of the stability criteria obtained using some of these methods are objectively compared with the simulation result from a series of numerical experiments. For each category of CF models discussed in this paper, several main factors that may affect the comparison results are considered. Among these factors, characteristics (i.e., intensity, duration, and magnitude) of disturbance appear in all three CF model categories, the platoon size is considered for B-CF models, response time for TD-CF models, and weights of leading vehicles for MAC-CF models. The impact of time delay and connectivity on stability region is also investigated. Main findings from the comparison analysis are:

- The inconsistency between the theoretical result and the simulation result indeed exists and varies with the factors considered in the study.
- A larger disturbance (in terms of intensity, duration, and magnitude) generally reduces the consistency between the theoretical result and the simulation result.

- For different platoon sizes with different disturbances, overall the theoretical result and the simulation result are consistent with each other nicely (above 90%), while the overall consistency is slightly deteriorated for platoons with a small number of vehicles (e.g., 10).
- Reaction time in TD-CF models significantly affects the consistency—both for the results of different reaction times and for the results of various disturbances with a fixed reaction time. Although in the human driving behaviour, local stability is always satisfied, our simulation result implies that a small desired time gap around 0.5s, which is often assumed for automated vehicles, is more likely to make the model locally unstable.
- Suboptimal weights assigned to different leading vehicles in MAC-CF models can cause the deterioration of the stability, and thus the loss of the effectiveness of connectivity.
- For CAV CF model, the time delay and connectivity both significantly affect the string stability, while they play totally opposite roles. Connectivity is critical to the maintenance of CAVs' stability in the presence of time delay.

Finally, based on the comprehensive review and the numerical experiments, main issues, challenges and research needs on stability analysis are discussed. Future research on CF models' stability analysis needs to be more application-oriented to increase stability analysis result's relevance to traffic operations and control in the real world. Particularly, the advent of connected and autonomous vehicles makes it possible for traffic operators to influence or control individual vehicle's movement by providing personalised information. How to design personalised information, or more generally, how to fully utilize the connected environment is a challenge. We believe that stability analysis can be a powerful tool in this regard. In future research, issues related to real traffic phenomena (e.g., different driving regimes, large perturbation caused by lane changing or hard breaking, etc.) need to be also considered to increase the practical value of CF models' stability analysis by making its result more application-ready.

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Appendix

Table A1. Representative studies of linear stability analysis

CF model	Study	Type of stability	Method
B-CF	Treiber and Kesting, 2013	Local; String	Characteristic equation/root extracting
	Wilson and Ward, 2011	Local; String	Laplace transform based method; Characteristic equation/root extracting
	Li et al., 2010	Local	Lyapunov creterion
	Li et al., 2013	Local; String	Characteristic equation/root extracting; Laplace transform based method; Lyapunov
	Chandler et al., 1958	String	Direct transfer function based method
	Orosz et al., 2010b	String	Characteristic equation/Hopf bifurcation; Laplace transform based method
	Orosz et al., 2011	String	Characteristic equation/Hopf bifurcation; Laplace transform based method
	Sau et al., 2014	String	Characteristic equation/root locus
TD-CF	Herman et al., 1959	Local	Laplace transform based method
	Zhang and Jarrett, 1997	Local; String	Characteristic equation/root extracting
	Kamath et al., 2015	Local	Characteristic equation/Nyquist
	Chandler et al., 1958	String	Direct transfer function based method
	Ferrari, 1994	String	Laplace transform based method
	Bando et al., 1998	String	Direct transfer function based method
	Orosz et al., 2010a	String	Characteristic equation/Hopf bifurcation
	Orosz et al., 2011	String	Characteristic equation/root extracting
	Ngoduy, 2013	String	Characteristic equation/root extracting
MAC-CF	Herman et al., 1959	String	Direct transfer function based method
	Lenz et al., 1999	String	Characteristic equation/Routh-Hurwitz
	Sau et al., 2014	String	Characteristic equation/root locus
	Ngoduy, 2015	String	Characteristic equation/root extracting
	Monteil et al., 2014	String	Characteristic equation/root extracting
	Swaroop and Hedrick 1996	String	Laplace transform based method
	Swaroop 1997	String	Laplace transform based method
	Liu and Goldsmith, 2001	String	Laplace transform based method
	Ge & Orosz, 2014	String	Laplace transform based method

Table A2. Summary of the major stability analysis methods

Method		Principles	Strengths	Issues
The characteristic equation based method	Root extracting	• Use exponential ansatz to represent perturbations • Calculate the real part of the root with the formula of extracting root	• Suitable for both the local and string stability analysis • Obtain the growth rate of perturbation directly • Efficient for simple models	• Computational complexity for complicated CF models
	The root locus method	• Use exponential ansatz to represent perturbations • Plot the roots on a complex plane	• Suitable for both the local and string stability analysis • Show stability graphically • Efficient for complicated CF models	• Need specific parameters • Only tell the trend of stability
	The Routh-Hurwitz criterion	• Use exponential ansatz to represent perturbations • Evaluate the real coefficients of the characteristic equation	• Suitable for both the local and string stability analysis • No need to solve equations • Efficient for simple CF models	• Not always effective for complicated characteristic equation
	The Nyquist criterion	• Use exponential ansatz to represent perturbations • Transform the quadratic characteristic equation into transfer function	• Useful for TD-CF models	• Only suitable for local stability analysis
	The Hopf bifurcation method	• Use exponential ansatz to represent perturbations • Set the real part of eigenvalues to be zero • Separate the real part and the imaginary part of the characteristic equation	• Suitable for both the local and string stability analysis	• Computational complexity for complicated CF models may lead to inaccurate results
Lyapunov criterion		• Construct an appropriate Lyapunov function to demonstrate the stability	• A good way to simplify the stability analysis • Nice extendibility for TD-CF models	• Need considerable skills to construct a Lyapunov function • Need to be analysed case by case • Not the sufficient and necessary condition
The direct transfer function based method		• Calculate the magnitude of transfer function with the Fourier ansatz	• A powerful method for string stability analysis • Efficient for TD-CF models	• Only suitable for string stability analysis • High complexity for cooperative CF models
The Laplace transform based method		• Calculate the magnitude of transfer function in the frequency domain	• A powerful method for both local and string stability analysis • Efficient for TD-CF models • Easy to construct transfer function	• High complexity for cooperative CF models